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Nonlinear Kac Model: Spatially Homogeneous Solutions and the Tjon Effect

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We study Kac's nonlinear model of the Boltzmann equation when the cross section $\sigma(\theta)$ does not satisfy the special symmetry condition $\sigma(\theta)=\sigma(\pi-\theta)$. We determine a differential system for the Laguerre moments of the odd and even velocity parts of the solutions. We consider the spatially homogeneous model in $1+1$ dimensions (velocity v and time t) when the even velocity part of the solution is provided by the Bobylev–Krook–Wu closed solutions and study the associated odd velocity part. We find that the solutions depend on the microscopic models of $\sigma(\theta)$. For one class of $\sigma(\theta)$, which has sums of exponential terms for the Laguerre moments, we establish the relations allowing the construction of the time-dependent solutions associated with any initial distribution. We find sufficient conditions on $\sigma(\theta)$ and on the even part such that the Laguerre series of the odd part converges. We establish a criterion for a well-defined linear combination of the moments cross section, and we check its validity for different numerical examples. We find that if the relaxation time for the even part is smaller than the corresponding one for the odd part and if the initial distribution has a narrow peak, then the Tjon effect exists for the complete B.K.W. solution (even + odd parts).

KEY WORDS: Nonlinear equations: Boltzmann equation; relaxation to equilibrium; spatially homogeneous Boltzmann equation; statistical physics; microscopic models of cross sections.

1. INTRODUCTION

In this paper (see also Ref. 1) we study the solutions of the nonlinear Kac model[2,3]

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$$
\begin{aligned}
&(\partial_t + v\partial_x) f(v, x, t) \\
&= \int_{-\pi}^{+\pi} \sigma(\theta) \int_{+\infty}^{+\infty} [f(v', x, t) f(w', x, t) - f(v, x, t) f(w, x, t)]\, dw\, d\theta \\
&v' = v\cos\theta - w\sin\theta, \qquad w' = v\sin\theta + w\cos\theta, \qquad \sigma(\theta) = \sigma(-\theta), \\
&\int_{-\pi}^{+\pi} \sigma(\theta)\, d\theta = \sigma_0 = 1
\end{aligned}
\tag{1.1}
$$

when the three variables v, x, t are reduced to only two: either v, t for the spatially homogeneous case or v, x for the stationary case.[1] The model, originally proposed by Kac[2] for $f(v, t)$, was extended by Uhlenbeck and Ford[3] for $f(v, x, t)$.

Here we are interested in the $f(v, t)$ case. Let us decompose f into its even $f^+(v, t)$ and odd $f^-(v, t)$ parts with respect to v. If in addition to the microscopic reversibility property $\sigma(\theta) = \sigma(-\theta)$, we assume the special symmetry $\sigma(\theta) = \sigma(\pi - \theta)$, [e.g., $\sigma(\theta) = \text{const.}$], then[4] the odd part becomes trivial: $f^- = f^-(v.0)e^{-t}$, without any link with the even part. However the Kac model is more interesting when the odd part is nontrivial. It has been shown by Ernst[4] that the even part has an exact solution, the so-called Bobylev–Krook–Wu[5,6] solution (hereafter called B.K.W. even mode). If $\sigma(\theta) \neq \sigma(\pi - \theta)$, then this even mode has a closed-form nontrivial odd partner and[7] a nontrivial solution $f = f^+ + f^-$ exists for the Kac model

$$
\begin{aligned}
&f^+(v, t) = \frac{\sqrt{b}}{2\pi} e^{-bv^2/2} \left[\frac{3-b}{2} + b(b-1)\frac{v^2}{2}\right], \\
&f^- = \frac{\sqrt{b}}{2\pi} e^{-bv^2/2} \frac{dvb}{\sqrt{2}} e^{-(\tau_0 - \tau_1)t}, \qquad b = (1 - ce^{-\sigma_2 t})^{-1} \\
&\tau_n = \int_{-\pi}^{+\pi} (\cos\theta)^n \sigma(\theta)\, d\theta, \qquad \sigma_{2n} = \int_{-\pi}^{+\pi} (\cos\theta \sin\theta)^{2n} \sigma(\theta)\, d\theta, \\
&\sigma_2 - \tau_1 + \tau_3 = 0, \qquad \tau_0 = 1, \qquad 0 < c < \tfrac{2}{3}
\end{aligned}
\tag{1.2}
$$

c and d being constants adjusted in such a way that $f(v, 0) > 0$. *We emphasize that the important new fact is the requirement of a condition on $\sigma(\theta)$ for the existence of the B.K.W. odd mode* (contrary to the B.K.W. even mode).

The removal of the assumptions $\sigma(\theta) = \text{const}$ or $\sigma(\theta) = \sigma(\pi - \theta)$ enlarges the class of solutions. Further it permits the introduction of microscopic conditions into the problem of the existence of the Tjon[8] effect. When this effect exists, it may produce at intermediate time a population of high-velocity particles larger than the one present at initial time or at equilibrium. The B.K.W. even mode alone cannot lead to the Tjon effect and consequently cannot represent a general feature of the solutions relaxing to equilibrium (other even solutions exhibit this effect). Adding just its odd partner, then the Tjon effect can exist.[7] Further, the importance of the effect depends on microscopic conditions. For the same initial conditions, $f(v, 0)$ the Tjon effect is present or absent depending on the model chosen for $\sigma(\theta)$.

In this paper we investigate whether or not these interesting properties are restricted to the odd B.K.W. mode or exist for a general class of nontrivial odd part, $f^-(v, t)$. We give up the search of exact solutions, do not retain the special symmetry $\sigma(\theta) = \sigma(\pi - \theta)$ and try to obtain a general formalism for the odd part f^-. A summary of the present results is given in a Note.[9]

In Section 2 we establish some general results for the $(1+1+1)$-dimensional problem $f(v, x, t)$: namely, the equations for the Laguerre moments $(-1)^n D_n^{\pm}$ when we expand f^+, $v^{-1} f^-$ into Laguerre polynomials $L_n^{\mp 1/2}$.

In Section 3 we come back to the spatially homogeneous case $f(v, t)$, assuming that f^+ is given by the B.K.W. even mode and investigate the possible associated odd part f^- or equivalently the Laguerre sets $(D_n^-(t))$. We first show that if the distribution function is positive at the initial time and if the even velocity part for all time values is positive (as is the B.K.W. even mode) then the distribution function is positive at all times.

Second we study the properties of the $D_n^-(t)$ obtained from integration of a linear differential system. For $n = 0, 1$ they are arbitrary: $D_n^- = d_n^- \exp(E_{0n} t)$, $E_{0n} = -1 + \tau_{2n+1} < 0$ and, for $n \geqslant 2$ they are recursively determined, integrating either at $t = 0$ or ∞:

$$D_n^-(t) = e^{E_{0n} t} \left[D_n^-(t_{\text{lim}}) + \int_{t_{\text{lim}}}^{t} \tilde{D}_n(t')\, dt' \right], \qquad t_{\text{lim}} = 0 \text{ or } \infty \tag{1.3}$$

$\tilde{D}_n$ is a linear combination of the D_q^-, $q \leqslant n-2$, with coefficients determined by the Laguerre moments of the B.K.W. even mode. D_n^- depends at most on n integration constants chosen among $d_q^- = D_q^-(0)$ or $\tilde{d}_q = \lim_{t \to \infty} [D_q^- \exp(-E_{0q} t)]$. The two representations are equivalent only if $\tilde{D}_n \to 0$ as $t \to \infty$ and this result does not hold for all $\sigma(\theta)$ models. The general solution is a superposition of simple solutions which are classified following the properties of a well-defined linear combination of the moments of $\sigma(\theta)$:

$$\beta_{Nn} = \tau_{2N+1} - \tau_{2n+1} - (n-N)\sigma_2, \qquad \sigma_2 = \tau_2 - \tau_4 \tag{1.4}$$

We obtain simple solutions characterized by two properties: all D_n^- *have only one time dependance* given mainly by an nth power of $e^{-\sigma_2 t}$ and there exists an infinite number of $d_n^- = 0$ which are computed by only *one or two parameters*. We find three classes of simple solutions that we call fundamental solutions: those with (i) $\beta_{0n} \neq 0$, $\forall n$, (ii) $\beta_{01} = 0$ (this family includes the B.K.W. odd mode, (iii) $\beta_{0N} = 0$ for some $N > 1$ value where D_n^- can include a product of t by an exponential. In case (i) the d_n^- are determined by one parameter and in the (ii) and (iii) cases by two parameters. Because, in general, we have no analytical guarantee that these fundamental solutions correspond to positive distributions, we cannot use them directly for physical application. However, the fundamental solutions are the building blocks of the physical solutions.[2]

In order to construct the physical solutions we restrict our study to $\beta_{0n} \neq 0$, $\forall n$ model. We give as input an infinite number of d_n^- such that $f(v, 0) > 0$ and we know that $f(v, t) > 0$. We determine the general explicit relations which give recursively all the parameters of the different exponential terms of the solution. These solutions which represent infinite mixing of class (i) fundamental solutions will be used for the study of the Tjon effect.

Third the construction of $\sigma(\theta)$ models corresponding to the different classes is given for very simple models.

At the end of Section 3 we establish the sufficient conditions in order that $N_2(t)$, the square of the norm of the solution f^-, built up with the Laguerre series, exists for any t value finite or infinite.

In the last (Section 4) we report numerical calculations of odd solutions associated with the B.K.W. even mode and study the Tjon effect. In analogy with what was done by Hauge and Praestgaard[11] for the Maxwell model with an even distribution, one can define a criterion explaining quite well the existence of the effect. Let us define the reduced distribution $F(v, t) = f(v, t)/f(v, \infty)$, assume that the odd part f^- has a nonzero Laguerre moment D_0^-, and compare, when both $|v|$, t are large, as a first approximation, the contributions coming only from $D_0^-(t)$, $D_2^+(t)$, the first odd and even Laguerre moments. One finds in this rough estimate that $F - 1$ is proportional to

$$F - 1 \simeq \frac{v}{\sqrt{2}} e^{(-t\tau_{\text{odd}})} D_0^-(0) + \frac{1}{2}\left(\frac{v^2}{2}\right)^2 e^{(-t\tau_{\text{even}})} D_2^+(0) \tag{1.5}$$

here $D_2^+(0) = -c^2$, $\tau_{\text{odd}} = \tau_0 - \tau_1$, $\tau_{\text{even}} = 2\sigma_2$. If $f^- \equiv 0$ or $D_0^-(0) = 0$, then $F - 1 < 0$ and there is no effect. If f^- is present, we note that its dominant behavior is small compared with that of f^+ (or $|v| \ll v^4$), we can think that

[2] See the analogy and the difference with the Maxwell-Bobylev even case, Ref. 10.

there is no chance for the Tjon effect. But if $\tau_{\text{odd}} < \tau_{\text{even}}$, and $|v|$, t large, the two terms can be comparable and $F-1$ can have a zero which moves as t increases. Let us define a criterion

$$\text{crit} = \tau_{\text{odd}} - \tau_{\text{even}} = \tau_0 - \tau_1 - 2\sigma_2 \tag{1.6}$$

then for $\text{crit} < 0$, $F-1$ can have a zero moving and we check this microscopic condition for different models. In an equivalent way we can speak in terms of relaxation times. If the relaxation time for the even part $(\tau_{\text{even}})^{-1}$ is smaller than the relaxation time for the odd part $(\tau_{\text{odd}})^{-1}$ then $F-1$ can have a zero moving. The importance of the effect depends also on the initial conditions and we check this property.

2. EQUATIONS FOR THE LAGUERRE MOMENTS

Here we look formally at the Laguerre moments for the odd and even parts of f. Taking into account $\sigma(\theta) = \sigma(-\theta)$ the equations for the full Kac model are

$$(\partial_t + \sigma_0 N_0^+) f^+(v) + v\partial_x f^-(v) = \int_{-\pi}^{+\pi} d\theta \int_{-\infty}^{+\infty} dw\, \sigma(\theta) f^+(v') f^+(w') \tag{2.1a}$$

$$(\partial_t + \sigma_0 N_0^+) f^-(v) + v\partial_x f^+(v) = \int_{-\pi}^{+\pi} d\theta \int_{-\infty}^{+\infty} dw\, \sigma(\theta) f^-(v') f^+(w') \tag{2.1b}$$

where $N_0^+ = \int_{-\infty}^{+\infty} f^+(v, x, t)\, dv$ is the local density, $\sigma_0 = \int_{-\infty}^{+\infty} \sigma(\theta)\, d\theta$, and $f^{\pm}(v)$ are simply written for $f^{\pm}(v, x, t)$.... We notice that the right-hand sides of (2.1a, b) are, respectively, quadratic in f^+ and linear in f^-.

Assuming that $\sigma(\theta) = \text{const}$, Kac has given expansions in terms of Hermite polynomials and deduced the system for the moments. Ernst,[4] in the homogeneous v, t case, with the help of the Fourier transform, has given the equations for the Hermite moments. Here, for the full v, t, x case, the gradient term $v\partial_x$ being present, we establish directly the equations for the Laguerre moments. We need, on the right-hand sides of (2.1a, b), a formula giving directly the integration of the product of two Laguerre polynomials in terms of Laguerre polynomials. Let us write

$$f^+(v, x, t) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} \sum_0^{\infty} (-1)^n D_n^+(x, t)\, L_n^{(-1/2)}\left(\frac{v^2}{2}\right) \tag{2.2a}$$

$$f^-(v, x, t) = \frac{\lambda}{\sqrt{2\pi}} v e^{-v^2/2} \sum_0^{\infty} (-1)^n D_n^-(x, t)\, L_n^{(1/2)}\left(\frac{v^2}{2}\right) \tag{2.2b}$$

λ being some normalization constant that we can choose at our convenience. We substitute the expansions (2.2a, b) into (2.1a, b). In order that the collision term reproduce Hermite polynomials, Kac has introduced the "Boltzmann bracket." Here for the right-hand sides of (2.1a, b) we need [especially for (2.1b)] the corresponding results for the Laguerre polynomials (see Section A1):

$$\int_{-\infty}^{+\infty}\left(\frac{v'}{\sqrt{2}}\right)^{\eta} e^{+w^2/2} L_{p'}^{(-1/2)}\left(\frac{w'^2}{2}\right) L_p^{(-1/2+\eta)}\left(\frac{v'^2}{2}\right)\frac{dw}{\sqrt{2}}$$
$$=\sqrt{\pi}\left(\frac{v}{\sqrt{2}}\right)^{\eta} C_{p+p'}^{p}(\cos\theta)^{2p+\eta}(\sin\theta)^{2p'} L_{p+p'}^{(-1/2+\eta)}\left(\frac{v^2}{2}\right) \qquad (2.3)$$

where $\eta=0$ for $L_p^{(-1/2)}$ or $\eta=1$ for $L_p^{(1/2)}$. It remains to expand the left-hand sides of (2.1a), (2.1b), respectively, in terms of $L_n^{-1/2}$, $L_n^{1/2}$. We find (Section A2)

$$\partial_t D_n^+ + 2\lambda\partial_x[(n+1/2)D_n^- + nD_{n+1}^-] = \sum_0^n D_q^+ D_{n-q}^+ C_n^q B_{qn} \qquad (2.4a)$$

$$\partial_t D_n^- + \lambda^{-1}\partial_x(D_n^+ + D_{n+1}^+) = \sum_0^n C_n^q D_q^+ D_{n-q}^- E_{qn} \qquad (2.4b)$$

with B_{qn}, E_{qn} defined by

$$B_{qn}=\int_{-\pi}^{+\pi}\sigma(\theta)\cos\theta^{2(n-q)}\sin\theta^{2q}\,d\theta, \qquad B_{0n}=\int_{-\pi}^{+\pi}\sigma(\theta)(\cos\theta^{2n}-1)\,d\theta$$
$$E_{qn}=\int_{-\pi}^{+\pi}\sigma(\theta)\cos\theta^{2(n-q)+1}\sin\theta^{2q}\,d\theta, \quad E_{0n}=\int_{-\pi}^{+\pi}\sigma(\theta)(\cos\theta^{2n+1}-1)\,d\theta \qquad (2.5)$$

In Eq. (2.4a), because $B_{00}=B_{01}+B_{11}$, the right-handside for $n=0, 1$ are zero. We find $\partial_t D_0^+ + \lambda\partial_x D_0^- = 0$ and $\partial_t D_1^+ + 2\lambda\partial_x(\frac{3}{2}D_1^- + D_0^-)=0$ which corresponds to conserved quantities.

3. ODD W.K.B. SOLUTIONS

3.11. General Considerations

We restrict our study to the spatially homogeneous case and show recursively that $f(v,t)>0$ *if*[3] *both* $f(v,0)>0$ *and* $f^+(v,t)>0$. Putting

[3] We believe that the first condition is sufficient (see the standard argument[12]), but are not aware of a rigorous proof.

$N_0^+ = 1$ in (2.1a, b) we obtain linear integral equations either for f of f^- with $f^+ e^{\sigma_0(t'-t)}$ as kernel

$$f(v, t) = e^{-\sigma_0 t} f(v, 0) + \int_0^t dt' \int_{-\pi}^{+\pi} d\theta\, \sigma(\theta) \int_{-\infty}^{+\infty} d\omega\, e^{\sigma_0(t'-t)} f^+(\omega', t') f(v', t')\, dt'$$

Iterating, we find $f(v, t) = \sum_0^\infty f^{(n)}(v, t)$ with $f(0) = e^{-\sigma_0 t} f(v, 0) > 0$,

$$f^{(n)}(v, t) = \int_0^t e^{\sigma_0(t'-t)}\, dt' \int_{-\pi}^{+\pi} \sigma(\theta)\, d\theta \int_{-\infty}^{+\infty} d\omega\, f^+(\omega', t') f^{(n-)}(v', t') > 0 \quad \text{if} \quad f^{(n-1)} > 0$$

In particular this positivity property holds if f^+ is the B.K.W. even mode (1.2).

All the proofs of this section are based on properties which are deduced from *the positivity of* $\sigma(\theta)$. We define $z = \cos\theta$ and study linear combination of moments τ_m which can be written $\int_{-\pi}^{+\pi} d\theta\, \sigma(\theta)\, g(\cos\theta)$ and have a definite sign if $g(\cos\theta)$ has a definite sign for $|(\cos\theta)| \leqslant 1$:

$$b_{p,n} = E_{0p} - (n-p)\sigma_2 < 0 \quad \text{for} \quad p < n, \qquad \sigma_2 = \tau_2 - \tau_4 \tag{3.1}$$

Due to $\sigma(\theta) > 0$, we have $\sigma_2 > 0$, $-E_{0p} = \tau_0 - \tau_{2p+1} > 0$

$$\begin{aligned}\beta_{p,n} &= E_{0p} - E_{0n} - (n-p)\sigma_2 \\ &= \tau_{2p+1} - \tau_{2n+1} - (n-p)(\tau_2 - \tau_4), \qquad p < n\end{aligned} \tag{3.2}$$

$$\beta_{p,n} - \beta_{p,n-1} = -\int \sigma(\theta)(1-z^2) z^2 (1 - z^{2(n-2)+1})\, d\theta < 0, \quad \forall n \geqslant 2$$

If $\beta_{p,p+2} < 0 \to \beta_{p,n} < 0$, $\forall n \geqslant 2$, $\forall n \geqslant p+2$

$$\beta_{p,p+2} = -\int \sigma(\theta)\, z^2 (1-z^2)(1 - z^{2(p-1)+1} + 1 - z^{2p+1})\, d\theta < 0, \quad \forall p \geqslant 1 \tag{3.3}$$

$\beta_{p,n<0}$ if $p \geqslant 1$, $n \geqslant p+2$

$$\beta_{0,n} = \int \sigma(\theta)(1-z^2)[z + z^3 + z^{2n-1} - nz^2] \text{ can vanishes}, \qquad \forall n \geqslant 1$$

$$\beta_{0,n+1} - \beta_{0,n} = -\int \sigma(\theta)(1-z^2)\, z^2 (1 - z^{2n-1})\, d\theta < 0, \qquad \forall n \geqslant 1 \tag{3.4}$$

If $\beta_{0,N} = 0 \to \beta_{0,n} < 0$, $\forall n > N$, $\forall N \geqslant 1$

$$\beta_{p,p+1} = -\int \sigma(\theta)\, z^2 (1-z^2)(1 - z^{2p-1}) < 0, \qquad \forall p \geqslant 1 \tag{3.5}$$

In Section 3.2 we study the properties of the odd Laguerre moments D_n^-, solutions of Eq. (2.4b), assuming that we know f^+ or the set (D_n^+). For $n=0, 1$, the moments D_n^- have the form $D_n^- = d_n^- e^{E_{0n} t}$ in which d_n^- are arbirary. For $n \geqslant 2$ we find these moments by integrating (2.4b). Let us define $\tilde{D}_n$ and introduce the integration constant d_n^- or $\tilde{d}_n$ at zero or infinity by

$$\tilde{D}_n = e^{-E_{0n} t} \sum_0^{n-2} D_{n-q}^+ D_q^- E_{n-q,n} C_n^q$$

$$D_n^-(t) = e^{E_{0n} t} \left[d_n^- + \int_0^t \tilde{D}_n(t')\, dt' \right] \tag{3.6}$$

$$D_n^-(t) = e^{E_{0n} t} \left[\tilde{d}_n + \int_\infty^t \tilde{D}_n(t')\, dt' \right] \tag{3.7}$$

Choosing for f^+ the B.K.W. even mode written down in Eq. (1.3) with $b = [1 - ce^{-\sigma_2 t}]^{-1}$ and substituting the corresponding $D_q^+ = (-c)^q (1-q) e^{-q\sigma_2 t}$ we find

$$\tilde{D}_n = \sum_{q=0}^{n-2} e^{-[(n-q)\sigma_2 + E_{0n}]t} \lambda_{qn} D_q^-(t) \tag{3.8}$$

with $\lambda_{qn} = (-C)^{n-q}(1-n+q) C_n^q E_{n-q,n}$. The validity of (3.7) requires $\tilde{D}_n \to 0$ as $t \to \infty$. If this property holds then $\tilde{b}_n = d_n^- + \int_0^\infty \tilde{D}_n(t)\, dt$. D_n^- depends upon n arbitrary constants, that we can choose from the d_p^-, $\tilde{d}_p$, $p \leqslant n-2$. Note that for $n=0, 1$ we have $d_n^- = \tilde{d}_n$. If we retain only one $\tilde{d}_n \neq 0$ or one $d_n^- \neq 0$, let us say for $n=N$, then we define a particular solution. However, for the solution d_N^- with $N>0$, f^- dominates over f^+ and f violates posivity for large v.

Let us choose $\tilde{d}_n \neq 0$ only for $n=N$ and iterate (3.7) for $n \geqslant N+2$. Taking into account (3.8), we can integrate up to ∞ only if $\beta_{Nn} < 0$. From the results (3.3) and (3.4) we can integrate for any $N>0$, and consequently $N=0$ gives a particular case of these solutions. These $\tilde{d}_N$ solutions have time dependance given by only one exponential term. Another simple family of solutions deduced from (3.7) is $\tilde{d}_n \neq 0$ for $n=0, 1$ and $\beta_{01}=0$. Indeed, because of (3.5), in that case, we have $\beta_{0n} < 0$ for $n \geqslant 2$, and these solutions depend on two parameters, but still one exponential time term for the $D_n^-(t)$. On the contrary if $\beta_{0N}=0$ for $N \geqslant 2$ then $\tilde{D}_N \to \text{const}$ as $t \to \infty$ and we cannot continue to use the representation (3.7) to seek simple solutions. If instead of (3.7), we start with (3.6), we can extend the existence of simple solutions for other β_{0N} values. We shall define as fundamental solutions the sets $(D_n^- = e^{(a+bnt)}(d_n^- + t\tilde{\tilde{d}}_n))$, which for all n values contain only one exponential time term.

In Section 3.2.1 we investigate the first possible D_n^- solutions of (3.6) for $n<4$. In Section 3.2.2 we define three types of fundamental solutions or of sets (D_n^-) for $n \geqslant N$ with $N \geqslant 0$ a fixed integer: (i) those for $\beta_{0N}=0\ \forall n$, (ii) those $\beta_{01}=0$, (iii) those with $\beta_{0N}=0$ for $N>1$. [In case (i) the solutions depend on only one parameter, d_N^-; on two parameters, d_0^-, d_1^-, in case (ii); and on d_0^-, d_N^- in case (iii).] Howwever, in general (except the B.K.W. odd mode), the associated solutions f^- cannot be written down in closed form and consequently we cannot control analytically the positivity of $f(v, 0)$. *A priori* it is not excluded that, similarly to the Maxwell–Bobylev case[5-10] all the fundamental solutions (except the B.K.W. odd mode) violate positivity. *Because we have no guarantee of the positivity of these solutions, they shall not be used for the construction of the physical solutions.* In Section 3.2.3 we construct the physical solutions which will be used for the study of the Tjon effect.

In Section 3.3 we give examples of $\sigma(\theta)$ models and in Section 3.4 we study the existence of the solutions $v^{-1}f^-$ in the Hilbert space spanned by the $e^{-v^2/2}L_n^{1/2}(v^2/2)$ orthogonal functions.

3.2.1. Explicit Solutions for the First D_n^- Moments. In order to avoid divergences when $t \to \infty$ we choose Eq. (3.6) starting at $t=0$. For $n<4$ and arbitrary initial data and $\sigma(\theta)$ we determine the first moments D_n^-. We discuss different possibilities coming from particular choices of d_p^- and $\sigma(\theta)$. The study is done in Appendix B1.

3.2.2. Determination of the Fundamental Solutions. The method is straightforward. We assume an ansatz for each family and relations that we verify for small n values, substitute into (3.6) and (3.7) for $q \leqslant n-2$. We identify the left- and right-hand sides, reconstruct D_n^-, and verify that both the ansatz and the relations still hold for general n. The study is done in Appendix B2.

3.2.3. General Solution with Only Pure Exponential Terms. We restrict our study to $\sigma(\theta)$ models such that $\beta_{0,n} \neq 0\ \forall n \geqslant 2$.

The general solution is a superposition of the solutions of class (i) and iterating Eq. (3.6) we easily see that it is of the type

$$D_n^- = e^{E_{0n}t} a_{0n} + \sum_{0}^{n-2} a_{n+1,n} e^{b_{m,n}t}, \qquad b_{m,n} = E_{0m} - (n-m)\sigma_2 \quad (3.9a)$$

Our aim is to determine the parameters $a_{m,n}$ from the initial conditions $f^-(v, 0)$ or equivalently from the set (d_n^-). We substitute the ansatz (3.9a) into Eq. (3.6) for all n, integrate (recall $\beta_{0n} \neq 0\ \forall n \geqslant 2$) and identify left- and right-hand sides. We find a linear system for the $a_{m,n}$:

$$a_{00} = d_0^-, \qquad a_{01} = d_1^-, \qquad a_{m+1,n}\beta_{m,n} = \lambda_{mn} a_{0m} + \sum_{q=m+2}^{n-2} \lambda_{qn} a_{m+1,q}$$

$$a_{0n} = d_n^- - \sum_{0}^{n-2} a_{m+1,n}, \qquad m = 0, 1,..., n-2 \tag{3.9b}$$

This very simple system for the $a_{m+1,n}$ can be solved either at n fixed or at m fixed. At *n fixed*, in the standard manner we determine recursively firstly $n = 2$, secondly $n = 3,..., n-1, n$. At *m fixed* we determine firstly $a_{1,n}$ and a_{02} from a_{00}:

$$a_{1n}\beta_{0,n} = \lambda_{0n} a_{00} + \sum_{q=2}^{n-2} \lambda_{qn} a_{1q}, \qquad a_{02} = d_2^- - a_{12}$$

secondly $a_{2,n}$ and a_{03} from a_{01}:

$$a_{2n}\beta_{1n} = \lambda_{1n} a_{01} + \sum_{q=3}^{n-2} \lambda_{qn} a_{2q}, \qquad a_{03} = d_3^- - a_{13} - a_{23}$$

thirdly $a_{3,n}$ and a_{04} from a_{02}:

$$a_{3n}\beta_{2,n} = \lambda_{2n} a_{02} + \sum_{q=3}^{n-2} \lambda_{qn} a_{3q}, \qquad a_{04} = d_4^- - a_{14} - a_{24} - a_{34}$$

and so on. This second method is more convenient for the numerical calculations. The general solution $D_n^-(t)$ has at most n arbitrary constants d_p^-, $p = 0, 1,..., n$, but this number can of course increase with n. On the contrary one can mix two, three... fundamental solutions of class (i) in such a way that the solution has only two, three... arbitrary constants. These mixings are obtained as particular cases of the general solution Eq. (3.9a, b). In Appendix B3 we study the mixing of two solutions N_1, N_2, $N_1 < N_2$ with two arbitrary constants $d_{N_1}^-$, $d_{N_2}^-$. The solution for $n \geqslant N_2$ is a sum of two exponential terms of the type $\exp[E_{0N_1} - (n - N_1)\sigma_2]t$ and $\exp[E_{0N_2} - (n - N_2)\sigma_2]t$. Further let us require that the time dependence has one term for each n value. For $n = N_2$ we must have $D_{N_2}^- = d_{N_2}^- \exp[E_{0N_1} - (N_2 - N_1)\sigma_2]t = d_{N_2}^-(\exp E_{0N_2} t)$ or $\beta_{N_1,N_2} = E_{0N_1} - E_{0N_2} - (N_2 - N_1)\sigma_2 = 0$. Owing to the results Eq. (3.3)–(3.5) we know that this is possible only for $\beta_{0,N_2} = 0$ and if we exclude $N_2 \geqslant 2$ which corresponds to class (iii) we see that the only possibilities are $\beta_{0,1} = 0$ or the class (ii) with two constant d_0^-, d_1^-.

3.3. Examples of $\sigma(\theta)$ Models

In Appendix B4 we construct very simple $\sigma(\theta)$ models corresponding to $\beta_{0n} \neq 0\ \forall n$, $\beta_{01}=0$ and $\beta_{0N}=0$ for $N>2$.

3.4. Existence Proof for the Odd Laguerre Series

We recall that $v^{-1}f^{-}$ is written as a sum of Laguerre polynomials and the square of the norm of the solution is

$$N_2(t) = \sum |D_n^{-}(t)|^2 \frac{\Gamma(n+3/2)}{\Gamma(n+1)}$$

The details are given in Appendix C; we obtain two main results. The most general one is the following:

$$N_2(t) < \infty \qquad \text{if } N_2(0) < \infty,\ |c| < 1, \text{ and } \int \sigma\,|\cos\theta|\, d\theta < \infty$$

We note that the constraints on the B.K.W. even mode, the $\sigma(\theta)$ and $f^{-}(v, 0)$, are very weak. However, the complete solution (even + odd) relaxes toward a Maxwellian as $t \to \infty$; there must exist another result saying that $N_2 \to 0$ when $t \to \infty$. We recall that $-E_{0n} > 0$; let us assume $\inf_n(-E_{0n}) \neq 0$; then one shows that $N_2(t) \to 0$ when $t \to \infty$.

4. ASYMPTOTIC BEHAVIOR, TJON EFFECT, AND NUMERICAL CALCULATIONS

4.1. Tjon Effect

An important property for the existence of the Tjon effect is the following. Let us call $v_{-}(0)$ [$v_{+}(0)$] the last negative [positive zero] of $F(v, 0) - 1$ where $F(v, t) = f(v, t)/f(v, \infty)$ is the reduced distribution function. If the effect exists, then as t increases the zero $v_{-}(t)$ [or $v_{+}(t)$] moves toward $-\infty$ (or $+\infty$). Consequently at intermediate times $0 < t < \infty$ we can have a population of high-velocity particles larger than the ones present (with the same velocity) at $t = 0$ or at equilibrium. The displacement of the zero will depend, as we shall see, on conditions on $\sigma(\theta)$. *However the effect really exists only if the* $F > 1$ *values are substantially larger than 1 and this last condition depends on the initial conditions on* $f(v, 0)$.

If the B.K.W. even mode is present alone, then the Tjon effect does not exist because we know[13] that $F < 1$ for $v^2/2 > 4$. Consequently, if the effect exists, it must result from a competition between even and odd parts $f^{\pm}$. In order to study the moving of the zero toward infinity, we shall define a criterion, using ideas similar to those presented by Hauge and Praestgaard[11] for even distributions alone. Let us retain in the Laguerre sums the contribution coming from the first terms D_0^-, $D_1^- L_1^{(+1/2)}$, $D_2^+ L_2^{-1/2}$. Consider t and $|v|$ large, $L_1^{+1/2} \simeq (-v^2/2)$, $L_2^{-1/2} \simeq (1/2)(v^2/2)^2$, and obtain the rough estimate

$$F-1 \simeq \frac{v}{\sqrt{2}}\left(d_0^- e^{-(\tau_0-\tau_1)t} + d_1^- e^{-(\tau_0-\tau_3)t}\frac{v^2}{2} + \cdots\right) - \frac{c^2}{2} e^{-2\sigma_2 t}\left(\frac{v^2}{2}\right)^2 + \cdots \tag{4.1}$$

where $D_2^+ = -c^2 e^{-2\sigma_2 t}$ for the B.K.W. even mode.

(i) Let us further neglect d_1^-, which means that we retain for $f^{\pm}$ only their first Laguerre terms D_0^-, D_2^+

$$F-1 \simeq \frac{|v|}{\sqrt{2}} e^{-(\tau_0-\tau_1)}\left[d_0^- \operatorname{sign}(v) - c^2 \frac{e^{(\tau_0-\tau_1-2\sigma_2)t}}{|v|\sqrt{2}}\left(\frac{v^2}{2}\right)^2\right] \tag{4.2}$$

If $d_0^- \operatorname{sign}(v) > 0$, the bracket in Eq. (4.1) will have a zero increasing when t increases if $\sigma(\theta)$ satisfies the criterion

$$\text{crit} = \tau_0 - \tau_1 - 2\sigma_2 < 0 \tag{4.3}$$

In order to study the validity of this crude criterion, we shall investigate classes of σ models (Section 3.3) where either the criterion has always the same sign or a transition occurs for which criterion changes sign. In both cases we shall check whether or not there is a moving of the last $F-1$ zero. For instance if $\sigma = (1/2)[\delta(\theta-\theta_1) + \delta(\theta+\theta_1)]$, $z = \cos\theta_1$, we find crit $= 0$ for $z \simeq 0.565$. For σ models of the Eq. (B.7) type and $\beta_{01} = 0$, the transition (crit $= 0$) occurs for $z_1 \simeq 0.707$. On the contrary for the models of Eqs. (B8) and (B9), crit > 0.

(ii) At higher order of approximation we can try to introduce contributions from other Laguerre moments. This is of course necessary if D_0^- or D_2^+ is zero. If $d_0^- = 0$, then Eq. (4.2) is replaced by

$$F-1 \simeq \frac{|v|^3}{2\sqrt{2}} e^{-(\tau_0-\tau_3)t}\left[d_1^- \operatorname{sign}(v) - \frac{c^2}{\sqrt{2}|v|^3}\left(\frac{v^2}{2}\right)^2 e^{(\tau_0-\tau_3-2\sigma_2)t}\right] \tag{4.2'}$$

$$\text{crit}' = \tau_0 - \tau_3 - 2\sigma_2, \text{ zero moving if crit}' < 0,\ d_1^- \operatorname{sign}(v) > 0 \tag{4.3'}$$

If $d_0^- \neq 0$, in the bracket of Eq. (4.2'), we have a supplementary term $e^{(\tau_1-\tau_3)t}(2/v^2) d_0^- \operatorname{sign}(v)$ that we must compare with the d_1^- term $[\tau_1 > \tau_3$ or $\tau_3 > \tau_1$, $\operatorname{sign}(d_0^- d_1^-)]$... and so on if we introduce other moments.

Our purpose here is to use the criterion Eq. (4.3) as a phenomenological tool in order to test the main features of the effect, although we are well aware that a more complete analysis must include other Laguerre moments.

Concerning the importance of the initial condition we remark that the B.K.W. even mode $f^+(v, 0)$ has two symmetric bumps and we shall see that the best effect is obtained when one of the two bumps disappears more or less.

4.2. Solutions Given by Initial Conditions

We choose the initial conditions [or the sets (d_n^-)] such that $f^-(v, 0)$ is a closed expression for which we can directly check the positivity of the sum $f^+(v, 0) + f^-(v, 0)$. We assume that the class of $\sigma(\theta)$ models belongs to class (i) where the associated solutions are superpositions of pure exponential terms. From the general formalism of Section 3.4 [Eq. (3.9a, b)] we can compute all the coefficients $a_{m,n}$, $b_{m,n}$ of the solution at $t \neq 0$. The simplest example is obtained by a sum of two exponentials:

$$f^-(v, 0) = \frac{v}{\sqrt{2}} \left[\sum L_n^{1/2} \left(\frac{v^2}{2} \right) \sum_{i=1}^{2} \mu_i C_i^n \right]$$
$$= \frac{v}{\sqrt{2}} \sum_{i=1}^{2} \frac{\mu_i}{(1 - C_i)^{3/2}} \exp \left[\frac{-v^2 C_i}{2(1 - C_i)} \right] \tag{4.4}$$

but of course other closed $f^-(v, 0)$ are easily constructed.

4.3. Numerical Calculations

Starting with solutions (d_n^-) such that $f^-(v, 0)$ are closed expressions and reconstructing the solutions at $t > 0$ with the general formalism [Eq. (3.9a, b)], we can at $t = 0$ both control the positivity $f(v, 0)$ and the convergence of the Laguerre series for f^-. Owing to the product of v by a finite number of Laguerre terms $\sum (-1)^n L_n^{1/2}(v^2/2) d_n^-$, the convergence becomes very poor when $|v|$ is large and we need 50 Laguerre terms for reproduce correctly the solution $|v| < 7$. For $t > 0$, the convergence is slightly better and works for larger $|v|$ values; this is due to the existence of exponential time-decreasing terms, but the previous problem is simply displaced to higher $|v|$ values. When the initial conditions correspond to the B.K.W. odd solution [$c_1 = c$, $c_2 = \mu_2 = 0$ in Eq. (4.4)] we can test the convergence of the Laguerre series for $t > 0$.

From our theoretical analysis, conditions on $\sigma(\theta)$ and $f^-(v, 0)$ control, respectively, the displacement of the last $(F - 1)$ zero and the possibility of substantial $F > 1$ values. The Tjon effect is obtained with the most favorable conditions on both $\sigma(\theta)$ and $f^-(v, 0)$. In order to test these ideas we consider $f^-(v, 0)$ given by (4.4) and two classes of σ models:

$$\text{(i)}\quad \sigma(\theta) = \frac{1}{2}\sum_{i=1}^{M} \lambda_i[\delta(\theta - \theta_i) + \delta(\theta + \theta_i)], \qquad \cos\theta_i = z_i,\ \tau_0 = 1 \quad (4.5)$$

For $M = 1$, the transition (crit $= 0$) occurs for $z_1 = \cos\theta_1 > 0.56$. In Fig. 1 (crit $= -0.242$) the conditions on $\sigma(\theta)$ and $f^-(v, 0)$ are favorable for both σ and f^-; in Fig. 2 (crit $= 0.331$) only for f^-; in Fig. 3 only for σ; and in Fig. 4 neither for f^- nor for σ. It is remarkable that in Fig. 3, where $f^-(v, 0)$ is very small we still observe the displacement of the zero [in accordance with the criterion of Eq. (4.3) which is independent of $f^-(v, 0)$] but the $F > 1$ values are very close to 1 and we conclude in that case that the effect does not exist. Now we still try to improve the σ and $f^-(v, 0)$ conditions. In Fig. 5 $f^-(v, 0)$ is the same as in Fig. 1 but $z_1 = 0.9$ instead of 0.75 far away from the transition value, although crit $= -0.207$ has a smaller modulus value. In Fig. 6a, b) where we observe the best effect, $z_1 = 0.9$ is the same as in Fig. 5 but we improve the $f^-(v, 0)$ conditions by choosing a narrow peak. For this $M = 1$ case the normalization $\sigma = 1$ fixes $\lambda = 1$ and we have only one parameter. The criterion Eq. (4.3), crit $= (1 - z_1)(1 - 2z^2 - 2z^3)$ has only one zero value: $z_{1\,\mathrm{crit}} \simeq 0.56$ and for $|z_1| < 1$ we observe the validity of the criterion for the moving of the $F - 1$ zero.

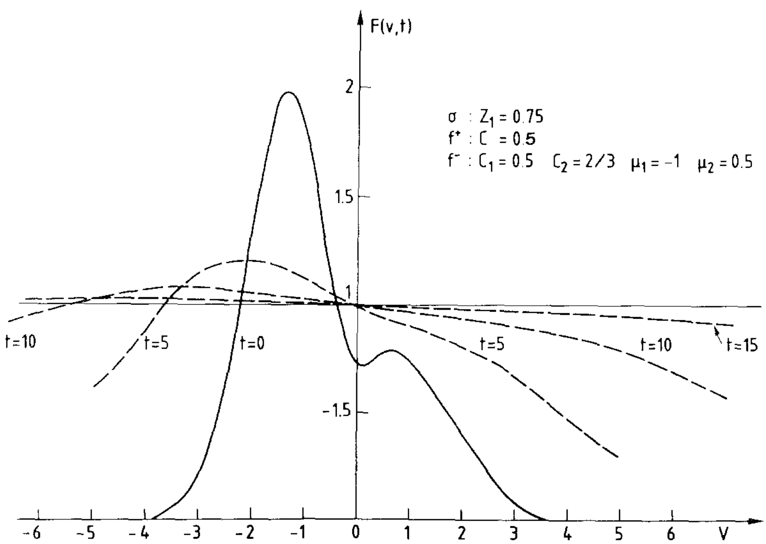


Fig. 1. Plot of $F(v, t)$ vs. v for $c = 0.5$; $c_1 = 0.5$, $c_2 = 2/3$, $\mu_1 = -1$, $\mu_2 = 0.5$ in Eq. (4.4), $z_1 = 0.75$ for $\sigma(\theta)$ in Eq. (4.5), crit $= -0.242$.

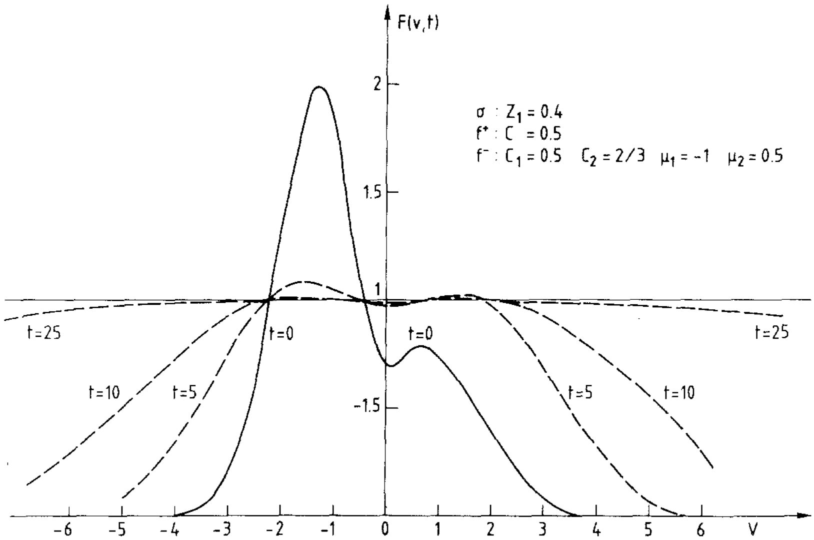


Fig. 2. The same as Fig. 1 but $z_1 = 0.4$, crit $= 0.331$.

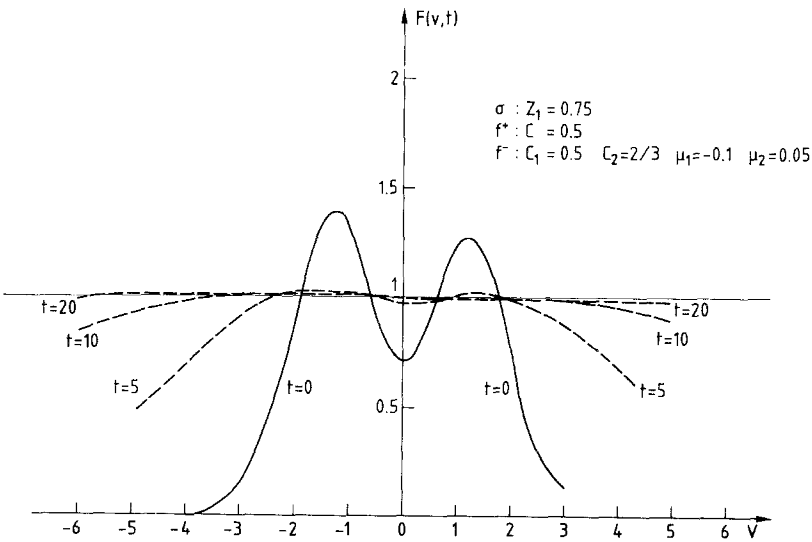


Fig. 3. The same as Fig. 1 but $\mu_1 = -0.1$, $\mu_2 = 0.05$, crit $= -0.242$.

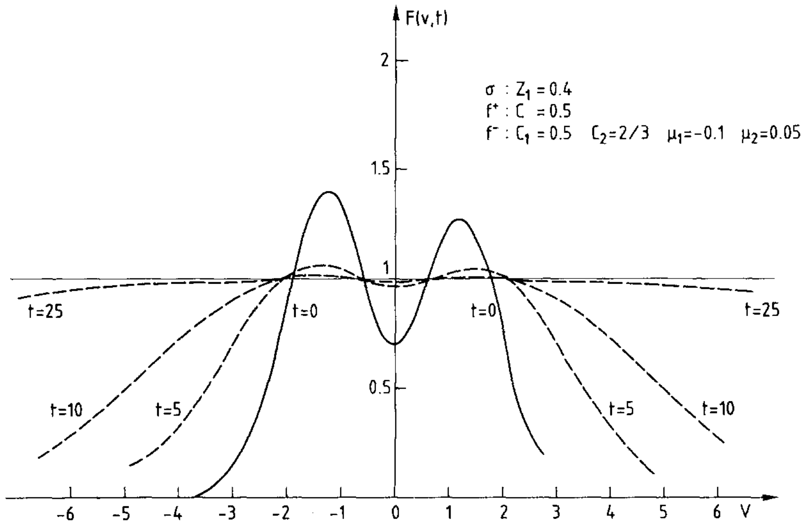


Fig. 4. The same as Fig. 1 but $\mu_1 = -0.1$, $\mu_2 = 0.05$, $z_1 = 0.4$, and crit $= 0.331$.

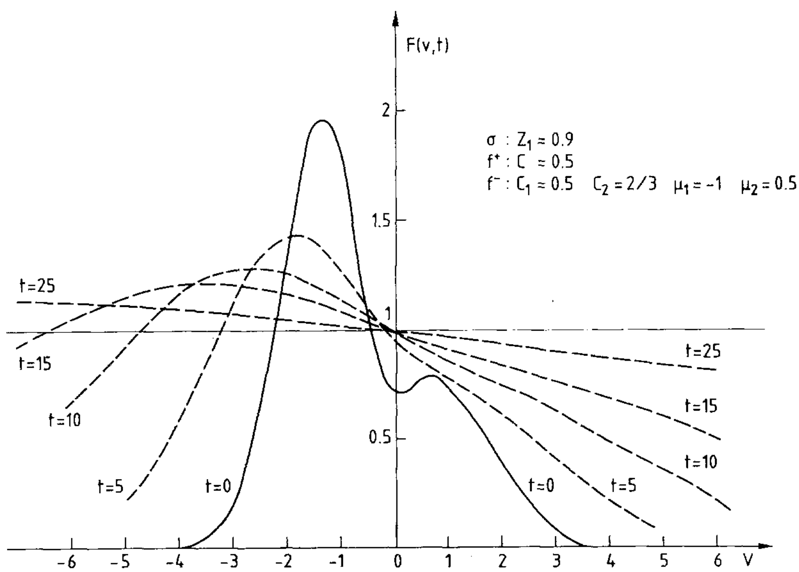
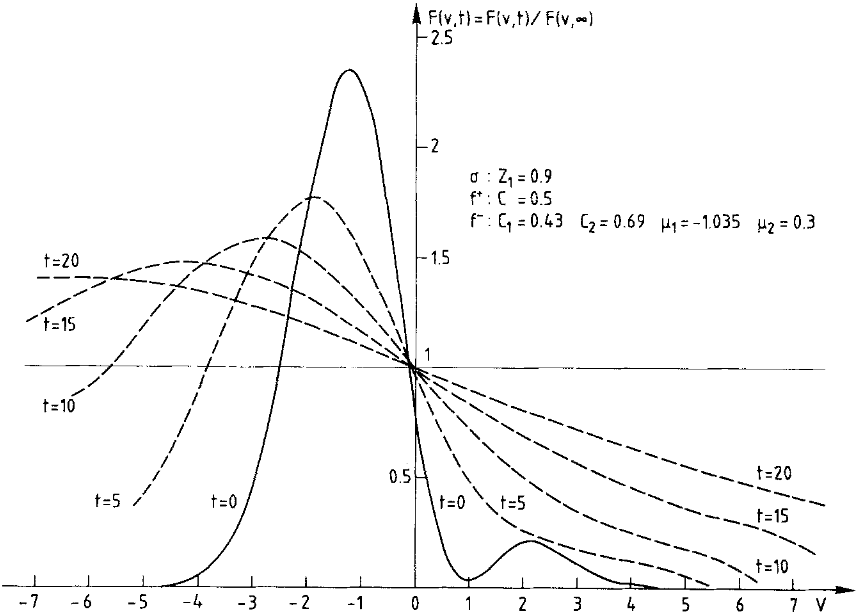
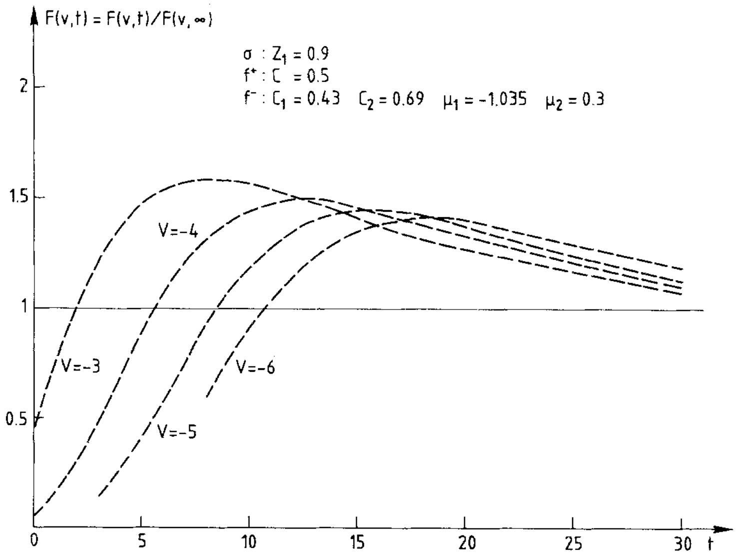


Fig. 5. The same as Fig. 1 but $z_1 = 0.9$, crit $= -0.207$.



(a)



(b)

Fig. 6. (a) Plot of $F(v, t)$ vs. v for $c = 0.5$; $c_1 = 0.43$, $c_2 = 0.69$, $\mu_1 = -1.035$ $\mu_2 = 0.3$, $z_1 = 0.9$, crit $= -0.207$. (b) The same as Fig. 6a but plot of $F(v, t)$ vs. t.

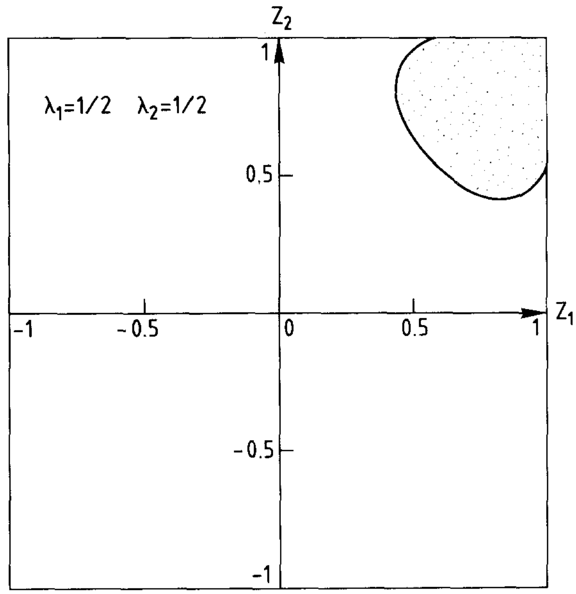


Fig. 7. Plot of the $|z_1| < 1$, $|z_2| < 1$ domain where crit < 0 (dotted region).

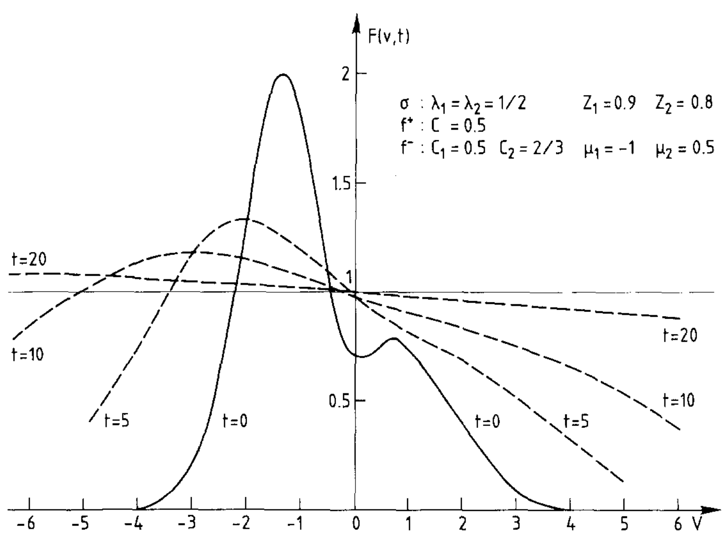


Fig. 8. Plot of $F(v, t)$ vs. v for $c = 0.5$, $c_1 = 0.5$, $c_2 = 2/3$, $\mu_1 = -1$, $\mu_2 = 0.5$ in Eq. (4.4); $z_1 = 0.9$, $z_2 = 0.8$, $\lambda_1 = \lambda_2 = 1/2$ for $\sigma(\theta)$, crit $= -0.2343$.

For $M=2$, we have three parameters λ_1, $\lambda_2=1-\lambda_1$, z_1, z_2. In Fig. 7 we choose $\lambda_1=\lambda_2$ and the dotted region represents the z_1, z_2 domain where

$$\text{crit}=\frac{1}{2}\sum_{i=1}^{2}(1-z_i)(1-2z_i^2-2z_2^3)<0$$

We still verify the moving of the $(F-1)$ zero, only in this dotted region, and in Fig. 8 for $z_1=0.9$, $z_2=0.8$ we see the Tjon effect:

$$\text{(ii)}\quad \sigma(\theta)=\frac{1}{4}\cos\frac{\theta}{2}\sum_{i=1}^{M}\bar{\lambda}_i(\cos\theta)^i, \qquad \tau_0=1 \tag{4.6}$$

For these smooth $\sigma(\theta)$ cross sections we have to take more terms, in order to observe the transition where the criterion changes sign and has appreciable negative values. For $M=1$, $\bar{\lambda}_1=1$, $\tau_0-\tau_1-2\sigma_2=26/63>0$ and in Fig. 9 we see that the $F-1$ zero is moving and though we choose for $f^-(v, 0)$ a favorable narrow peak, there is no effect. For $M=2$, $\bar{\lambda}_2=3(1-\bar{\lambda}_1)$, we have one parameter $\bar{\lambda}_1$. $\sigma(\theta)>0$ requires $0.75\leqslant\bar{\lambda}_1\leqslant 1.5$ and in this interval the values of the criterion remain positive. We observe neither the displacement of the zero, nor the effect. For $M=3$, $(7/15)\bar{\lambda}_3=1-\bar{\lambda}_1-\bar{\lambda}_2/3$ and we have (two parameters. $\sigma(\theta)>0$ gives a domain into the

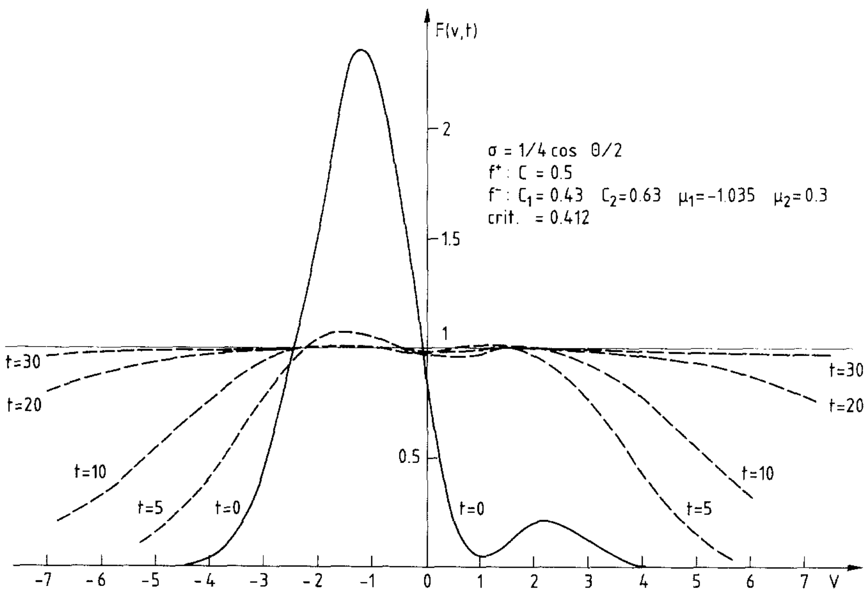


Fig. 9. The same as Fig. 8 but $\sigma(\theta)=(1/4)\cos\theta/2$ in Eq. (4.6), crit $=0.412$.

$\bar{\lambda}_1, \bar{\lambda}_2$ plane. Finally the vanishing of crit $= \bar{\lambda}_1(26/63) - (743/30(105)]\bar{\lambda}_2 + \bar{\lambda}_3[3986/(33)(13)]$ gives two subregions *where* crit > 0 *or* < 0 (dotted region in Fig. 10). In both subdomains we do not observe the Tjon effect if the zero moves, it moves slowly but still more slowly when crit > 0. Let us consider the same $f^-(v, 0)$ as in Fig. 9 and compare the zero $v_-(t)$ for two cases: $\bar{\lambda}_1 = 0.2$, $\bar{\lambda}_2 = 0.9$, crit $= -0.0179$ and $\bar{\lambda}_1 = 0.32$, $\bar{\lambda}_2 = 0.9$, crit $= 0.0116$. In both cases $v_-(0) \simeq -2.2$, whereas $v_-(50) = -4.4$ when crit < 0 and $v_-(50) = -3.2$ when crit > 0. The drawback of the criterion is explained here by the fact that the modulus values of the crit < 0 are small, less than 0.018. For $M = 4$, we have one more parameter at our disposal and we find larger negative values for the criterion. We observe both the moving of the $F - 1$ zero and the Tjon effect [see Fig. 11 with a narrow $f(v, 0)$ peak].

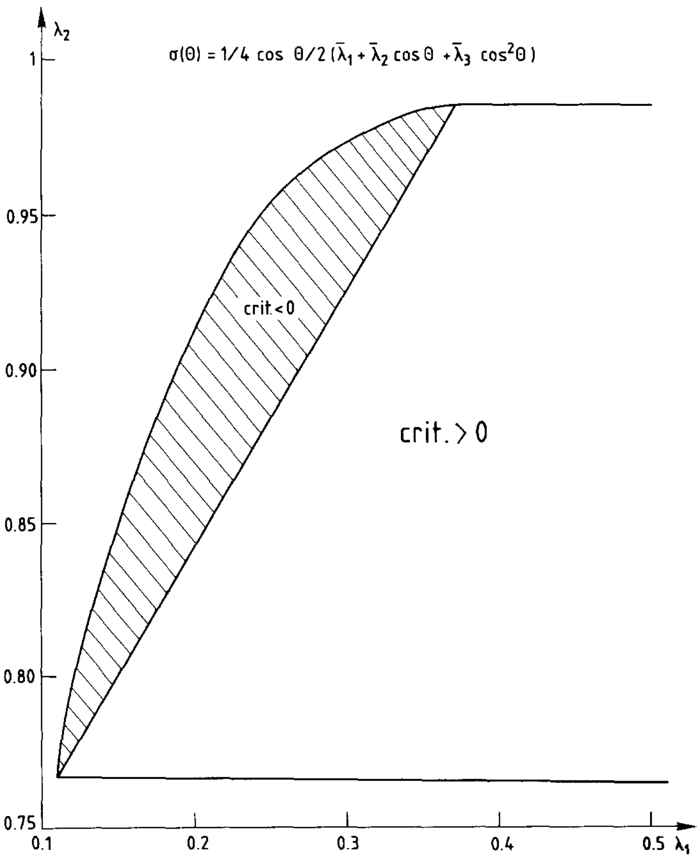


Fig. 10. Plot of the $\bar{\lambda}_1, \bar{\lambda}_2$ region where $\sigma(\theta) > 0$, $\tau_0 = 1$ and crit < 0 (dotted region).

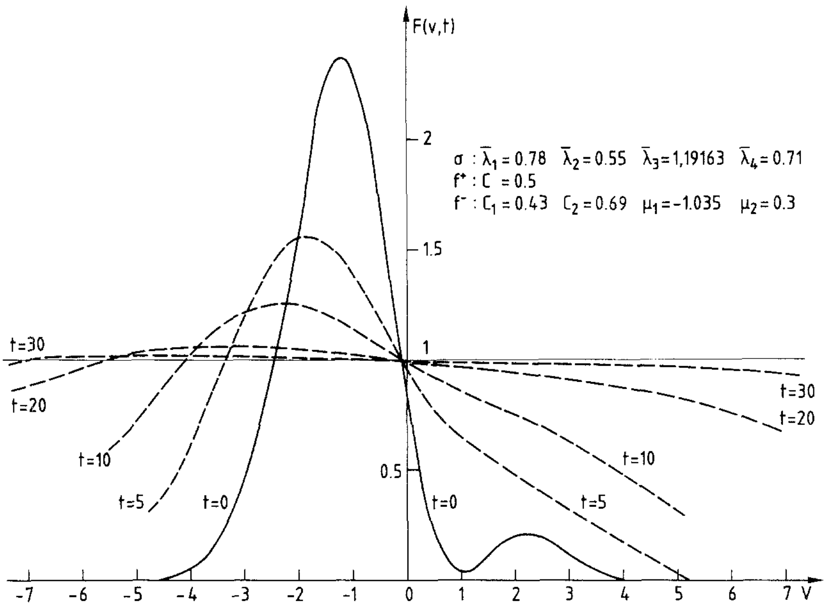


Fig. 11. The same as Fig. 9, but $\lambda_1 = 0.078$, $\lambda_2 = 0.5$, $\lambda_3 = 1.1916$, for $\sigma(\theta)$ in Eq. (4.6), crit $= -0.075$.

4.4. Linearized Versus Nonlinear Formalisms

In the linearized version of the Kac model, the Laguerre moments are: $D_0^+ = 1$, $D_1^+ = 0$, $D_2^+ = d_2^+ e^{-2\sigma_2 t}$, $D_3^+ = d_3^+ e^{-3\sigma_2 t}$, $D_n^+(t) = d_n^+ \exp[(B_{0n} + B_{nn})t]$; $D_n^-(t) = d_n^- e^{E_{0n} t}$. For the first moments D_n^+, $n \leqslant 3$, D_n^-, $n \leqslant 1$, these moments are identical with those of the nonlinear formalism. Consequently the discussion of the asymptotic behavior of the solutions, provided only these moments occur, is the same in both formalisms (for instance the definition of crit and crit′).

However let us remark that while the positivity of $\sigma(\theta)$ alone allows in both formalisms to prove that the D_n^+, $n > 3$ decrease faster than D_3^+, a similar property for D_n^- compared with D_0^-, D_1^- is not so obvious [owing to the nonpositivity of the odd $\sigma(\theta)$ moments which enter into E_{0n}]. More explicitly one can prove (similarly to what was done for the Boltzmann equation with Maxwell interaction[13]) that the full nonlinear D_n^+ decrease at least like $\exp(B_{0n} + B_{nn})t$, i.e., the linearized ones; and secondly (Appendix D) for $n > 3$, $\exp(B_{0n} + B_{nn})t$ decrease more than $\exp(-3\sigma_2 t)$. On the contrary for the odd part, in the linearized case, $D_n^-/D_0^- = (d_n^-/d_0^-)\exp[-(\tau_1 - \tau_{2n+1})t]$ and the moments $\tau_1 - \tau_{2n+1}$ do not have a definite sign.

5. CONCLUSION

The aim of this paper was twofold. First we wanted to show that the closed B.K.W. odd mode[7] was not the only nontrivial solution f^+ that we can add to the B.K.W. even mode f^+. We have developed a general formalism that constructs time-dependent $f^-(v, t)$ from any $f^-(v, 0)$. We have seen that the condition $\tau_1 - \tau_3 - \sigma_2 = 0$, necessary for the B.K.W. odd mode, was not essential, but the crucial constraint is the microscopic one, $\sigma(\theta) \neq \sigma(\pi - \theta)$. Let us notice that the special symmetry $\sigma(\theta) = \sigma(\pi - \theta)$ has no physical basis in a one-dimensional velocity model, like the Kac model, and that it was previously introduced[4] in order to simplify the formalism.

Secondly we wanted to verify that the particular properties[7] found for the Tjon effect[8] were not restricted to the B.K.W. odd mode. The B.K.W. even mode alone, f^+, cannot exhibit this effect and consequently cannot represent a general feature for the relaxation to equilibrium of the Boltzmann equation. At least for the Tjon effect this drawback disappears for the complete solution $f^+ - f^-$ if we add for f^- either the B.K.W. odd mode or the odd solutions f^- studied in this paper. More generally, from this paper, it is clear that this effect cannot be well understood without the introduction of the odd part f^- into the discussion. Let us briefly recall the Hauge–Praestgaard[11] argument for an even velocity distribution $f^+(v, t)$ alone. For the reduced distribution $F^+(v^2, t) = f^+(v^2, t)/f^+(v^2, \infty)$, concerning the relaxation toward equilibrium one can, in a rough estimate, retain the contribution of the first Laguerre moment $D_2^+(t) = d_2^+ e^{-2\sigma_2 t}$ and for v^2, t large obtain $F^+(v^2, t) - 1 \simeq (d_2^+/2) e^{-2\sigma_2 t} |v^2/2|^2$. Only two possibilities can occur: either $d_2^+ > 0$, $F > 1$ and we have the Tjon effect or $d_2^+ < 0$ and we have no effect (as is the case of instance for the B.K.W. even mode). Of course, in particular cases, corrections to this criterion can be necessary; for instance the third Laguerre moment can be the dominant one.[13] In general it is difficult to justify mathematically the Hauge–Praestgaard criterion; however, it is a very good phenomenological tool.[13] Let us now introduce the odd part f^- and similarly retain the first odd Laguerre moment $d_0^- e^{-(\tau_0 - \tau_1)t}$. For t, $|v|$ large we find

$$F(v, t) - 1 \simeq \frac{|v|}{\sqrt{2}} e^{-\tau_{\text{odd}} t} \left[\operatorname{sign}(v) d_0^- + \frac{d_2^+}{\sqrt{2}} \frac{1}{|v|} \left| \frac{v^2}{2} \right|^2 e^{(\tau_{\text{odd}} - \tau_{\text{even}})t} \right]$$

with $\tau_{\text{odd}} = \tau_0 - \tau_1$, $\tau_{\text{even}} = 2\sigma_2$. We define $\text{crit} = \tau_{\text{odd}} - \tau_{\text{even}} = \tau_0 - \tau_1 - 2\tau_2$ and we have three possibilities:

(i) $\text{crit} > 0$, $d_2^+ > 0$, then $F > 1$ and we have the Tjon effect.

(ii) $\text{crit} > 0$, $d_2^+ < 0$, $F < 1$ and we have no Tjon effect.

(iii) $\text{crit} < 0$, then $F - 1$ will have a zero $v(t)$ moving along the v axis $\operatorname{sign}(v) d_0^+ d_2^+ < 0$. The effect will exist if this zero, at $t = 0$, is at the border of a sufficiently narrow peak such that for $t>0$ the peak will spread out with appreciable $F>1$ values. [Of course it can happen that $d_0^- = 0$ or that the dominant contribution is given by $D_1^-(t)$ and we must consider $\tau_0 - \tau_3 - 2\sigma_2, \ldots$ and so on.]

In this paper, f^+ is the B.K.W. even mode, $d_2^+ = -c^2$ and we have only verified the possibilities (ii) and (iii) for $d_2^+ < 0$. It remains to check the correctness of this analysis, of the Tjon effect, for $f^+(v, t)$ solutions with $d_2^+ > 0$.

If we introduce the relaxation times: $T_{\text{even}} = (\tau_{\text{even}})^{-1}$ and $T_{\text{odd}} = (\tau_{\text{odd}})^{-1}$ of the even and odd parts of $F-1$ the above discussion can be repeated. If T_{odd} is smaller than T_{even} then the even part always dominates and the effect exists or not depending whether d_2^+ is positive or negative. If T_{odd} is larger than T_{even} then there exists times t and velocities v where the odd and even contributions are comparable and the Tjon effect exists if the initial conditions are favorable or equivalently if the distribution has a narrow peak at $t=0$.

Let us recall that in the conservation laws $\int f\,dv$ and $\int fv^2\,dv$, only $f^+(v, 0)$ contributes, both $f^-(v, 0)$ and $\sigma(\theta)$ are not present. However, for a given $f^+(v, 0)$, in the possibility (iii), we can always manage $f^-(v, 0)$ and $\sigma(\theta)$ in order to find a narrow peak for $f(v, 0)$ and the constraint $\tau_0 - \tau_1 - 2\sigma_2 < 0$. *From the above analysis of the different possibilities, then for any given $f^+(v, 0)$ we can always find macroscopic conditions of $f^-(v, 0)$ and microscopic conditions on $\sigma(\theta)$ such that the Tjon effect exists.*

Finally we notice that the discussion of the asymptotic behaviors of the solution (existence or not of the effect) occurs with the first moments D_2^+, D_3^+, D_0^-, D_1^-. All these four moments, satisfying linear differential equations, are identical with those obtained from the linearized version of the Kac model. Consequently the discussion and the criterion are the same in both nonlinear and linearized formalisms. However, owing to the non-positivity of the odd moments of $\sigma(\theta)$, the comparison for higher odd Laguerre moments is not simple.

APPENDIX A

A1. We want to prove the formula

$$\int_{-\infty}^{\infty} \left(\frac{v'}{\sqrt{\mu}}\right)^{\eta} e^{-w^2/\mu} L_{p'}^{(-1/2)}\left(\frac{w'^2}{\mu}\right) L_p^{(-1/2+\eta)}\left(\frac{v'^2}{\mu}\right) \frac{dw}{\sqrt{\mu}}$$
$$= \sqrt{\pi} \left(\frac{v}{\sqrt{\mu}}\right)^{\eta} \frac{(p+p')!}{p!\,p'!} (\cos\theta)^{2p+\eta} (\sin\theta)^{2p'} L_{p+p'}^{(-1/2+\eta)}\left(\frac{v^2}{\mu}\right) \quad \text{(A1)}$$

$\mu > 0$, $\eta = 0$ or 1, $v' = v\cos\theta - w\sin\theta$, $w' = v\sin\theta + w\cos\theta$.

Using the classical results for the Laguerre $L_n^{\pm 1/2}$ and Hermite polynomials

$$(-1)^p H_{2p+\eta}(x) = x^\eta 2^{2p+\eta} L_p^{(-1/2+\eta)}(x^2) \tag{A2}$$

$$H_q(\alpha_1 x_1 + \alpha_2 x_2) = q! \sum_{p=0}^{q} \alpha_1^p \alpha_2^{q-p} H_p(x_1)\, H_{q-p}(x_2), \qquad \alpha_1^2 + \alpha_2^2 = 1 \tag{A3}$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_p(x)\, H_{p'}(x)\, dx = \pi^{1/2} 2^p p!\, \delta_{p,p'} \tag{A4}$$

$$\sum_{r=0}^{\min(p,p')} r!(-2)^r C_p^r C_{p'}^r H_{p-r}(x)\, H_{p'-r}(x) = H_{p+p'}(x) \tag{A5}$$

With the help of (A2) and (A3) we find for the Laguerre polynomials

$$\left(\frac{v'}{\sqrt{\mu}}\right)^\eta L_p^{(-1/2+\eta)}\left(\frac{v'^2}{\mu}\right) = \frac{(-1)^p}{2^{2p+\eta}} \frac{(2p+\eta)!}{p!} \sum_{p_1=0}^{2p+\eta} \frac{(\cos\theta)^{p_1}(-\sin\theta)^{2p+\eta-p_1}}{(p_1!)(2p+\eta-p_1)!}$$
$$\times H_{p_1}\left(\frac{v}{\sqrt{\mu}}\right) H_{2p+\eta-p_1}\left(\frac{w}{\sqrt{\mu}}\right) \tag{A6}$$

$$L_{p'}^{(-1/2)}\left(\frac{w'^2}{\mu}\right) = \frac{(-1)^{p'}}{2^{2p'}} \frac{(2p')!}{(p')!} \sum_{p_1'=0}^{2p'} \frac{(\sin\theta)^{p_1'}(\cos\theta)^{2p'-p_1'}}{(p_1')!\,(2p'-p_1')!}$$
$$\times H_{p_1'}\left(\frac{v}{\sqrt{\mu}}\right) H_{2p-p_1'}\left(\frac{w}{\sqrt{\mu}}\right) \tag{A7}$$

In order to obtain the left-hand side of (A1) we multiply both (A6) and (A7) by $e^{-w^2/\mu}\mu^{-1/2}$, integrate over w from $-\infty$ up to $+\infty$, and the orthogonality property for the Hermite polynomials gives the restriction $2p' - p_1' = 2p + \eta - p_1 = p_2$. Consequently the left-hand side of (A1) is

$$\left(\frac{\cos\theta}{2}\right)^{2p+\eta} \left(\frac{\sin\theta}{2}\right)^{2p'} \frac{\sqrt{\pi}}{(p')!\,p!} \left[\sum_{p_2=0}^{\inf(2p+\eta,2p')} (-2)^{p_2} C_{2p+\eta}^{p_2} C_{2p'}^{p_2} H_{2p+\eta-p_2}\right.$$
$$\left.\times\left(\frac{v}{\sqrt{\mu}}\right) H_{2p'-p_2}\left(\frac{v}{\sqrt{\mu}}\right)\right] \frac{(-1)^{p+p'}}{2^{2p+2p'+\eta}} \tag{A8}$$

From (A5) we see that the bracket is just $H_{2p+2p'+\eta}(v/\sqrt{\mu})$ and, with the help of (A2), (A8) is just the left-hand side of (A1).

A2. We deduce from Eqs. (2.1a, b) the system for the $D_n^\pm$ when

f^+, vf^- satisfy the expressions (2.2a, b). With the help of (A1) we find for the right-hand side of Eq. (2.1a):

$$\frac{e^{-v^2/2}}{(2\pi)^{1/2}} \sum_n L_n^{(-1/2)}(-1)^n \sum_{q=0}^{n} D_q^+ D_{n-q}^+ C_n^q \int \sigma(\theta) \cos \theta^{2q} (\sin \theta)^{2(n-q)} \, d\theta \qquad \text{(A9)}$$

For the second term at the left-hand side of (2.1a) we use the relation $xL_n^{1/2}(x) = (n+3/2)\, L_n^{(-1/2)}(x) - (n+1)\, L_{n+1}^{(-1/2)}(x)$ so that the whole left-hand side reads

$$\frac{e^{-v^2/2}}{(2\pi)^{1/2}} \sum_n (-1)^n L_n^{(-1/2)} \left[(\partial_t + \sigma_0 D_0^+) D_n^+ + 2\lambda \partial_x \left(\left(n + \frac{3}{2} \right) D_n^- + D_{n-1}^- \right) \right]$$

With the help of (A1) we find for the right-hand side of Eq. (2.1b)

$$\frac{\lambda v e^{-v^2/2}}{(2\pi)^{1/2}} \sum_n (-1)^n L_n^{(1/2)} \sum_{q=0}^{n} C_n^q D_{n-q}^- D_q^+ \int \sigma(\theta) \cos \theta^{2(n-q)+1} \sin \theta^{2q} \, d\theta$$

For the second term at the left-hand side of (2.1b) we use $L_n^{(-1/2)}(x) = L_n^{1|2}(x) - L_{n-1}^{1/2}(x)$ and obtain for the whole left-hand side

$$\frac{v e^{-v^2/2}}{(2\pi)^{1/2}} \sum_n (-1)^n L_n^{1/2} [(\partial_t + \sigma_0 N_0^+) D_n^- \lambda + \partial_x (D_n^+ + D_{n+1}^+)]$$

APPENDIX B

B1. First D_n^- Moments. As n increases the number of different time-dependant terms for D_n^- increases too. However, for particular initial data and $\sigma(\theta)$, D_n^- has only one term, which turns out to be essentially a power of $e^{-\sigma_2 t}$. We use it with Eq. (3.6) starting at $t = 0$. If $D_0^- = 0$, $D_1^- = 0$, they are arbitrary while every other moment D_n is recursively determined from D_q^-, $q = 0, \dots, n-2$. If $D_0^- = 0$, then D_1^- and D_2^- are arbitrary, and so on. For $n = 0$, 1 we have $d_0^- e^{-E_{00} t}$ and $d_1^- e^{-E_{01} t}$, which are the first terms of family (i) for $N = 0$ and 1. If we want to obtain the second term of (ii) we must require $D_1^- = e^{(E_{00} - \sigma_2) t} d_1^-$ or the relation $\beta_{0,1} = E_{00} - \sigma_2 - E_{01} = 0$ which from Eq. (3.4) is possible. For $n = 2$ we have two possibilities:

$$D_2^- = e^{E_{02} t} \left[d_2^- - \frac{\lambda_{02} d_0^-}{\beta_{0,2}} \right] + e^{b_{0,2} t} \frac{\lambda_{0,2}}{\beta_{0,2}} d_0^-, \qquad \beta_{0,2} \neq 0$$

$$D_2^- = e^{(E_{00} - 2\sigma_2) t} [d_2^- + t \tilde{\tilde{d}}_2] \; \tilde{\tilde{d}}_2 = \lambda_{02} d_0^-, \qquad \beta_{0,2} = 0$$

(a) $\beta_{02} \neq 0$. The general solution has two terms but if we require $d_2^- = (\lambda_{0,2}/\beta_{0,2}) d_0^-$ then $D_2^- = e^{(E_{00} - 2\sigma_2)t} d_2^-$. If further $d_1^- = 0$ we find the two first terms of family (i) with $N = 0$ (if $d_1^- \neq 0$, then D_3^- has two different times terms except in $\beta_{01} = 0$). If further $d_1^- \neq 0$ and $\beta_{0,1} = 0$, then $\beta_{0,2} \neq 0$ and we have the three first terms of family (ii).

(b) $\beta_{0,2} = 0$, then necessarily $\beta_{0,1} \neq 0$ [see Eq. (3.4)] and we have the two first terms of family (iii) if $d_1^- = 0$.

For $n = 3$, recalling that $\beta_{1,3} \neq 0$, we still have two possibilities

$$D_3^- = e^{E_{03}} \left(d_3^- - \sum_{m=0}^{1} \frac{\lambda_{m3} d_m^-}{\beta_{m,3}} \right) + \sum_{m=0}^{1} \frac{e^{b_{m3}t} d_m^-}{\beta_{m,3}} \qquad \text{for } \beta_{03} \neq 0$$

$$D_3^- = e^{(E_{00} - 3\sigma_2)t} \left(d_3^- - \frac{\lambda_{1,3} d_1^-}{\beta_{1,3}} + t\lambda_{0,3} d_0^- \right) + e^{(E_{01} - 2\sigma_2)t} \frac{\lambda_{1,3} d_1^-}{\beta_{1,3}} \qquad \text{for } \beta_{03} = 0$$

(a) $\beta_{0,3} \neq 0$. The general solution has three terms but if we require $d_3^- = \sum_0^1 (\lambda_{m3} d_m^- / \beta_{m3})$ then $D_3^- = \sum_0^1 \exp(b_{m3} t)(\lambda_{m3} d_m^- / \beta_{m3})$. If further $d_1^- = 0$, then $D_3^- = \exp[(E_{00} - 3\sigma_2)t] d_3^-$ a number of the family (i) with $N = 0$. If $d_1^- \neq 0$ but $\beta_{01} = 0$, then $\beta_{03} \neq 0$ and $b_{13} = E_{00} - 3\sigma_2$; in that case $D_3^- = \exp[(E_{00} - 3\sigma_2)t] d_3^-$, a member of the family (ii).

(b) $\beta_{03} = 0$ which leads to $b_{13} = E_{01} - 2\sigma_2$. If $d_1^- = 0$ we have a member of family (iii).

B2. Fundamental Solutions. We seek the particular solutions $[D_n^-(t)]$ which for every n have only one time-dependant term. It turns out that the time appears mainly as a power of $e^{-\sigma_2 t}$.

First we consider (3.7), the integral equation with $\tilde{d}_n$ as integration at infinity. We study the simple solutions $\tilde{d}_n = 0$ except for $n = N$, N being either 0 or a fixed integer. In (3.7) the first $D_n^- \neq 0$ appears for $n = N + 2$ and the integration exists when $t' \to \infty$ if $\beta_{N\,N+2} < 0$. From the relations (3.2)–(3.5) we know for $N > 0$ that $\beta_{Nn} < 0$ for $n > N$. For $N = 0$ we must assume that $\beta_{02} < 0$, from (3.2)–(3.5) then the other β_{0n} are negative for $n > 2$ values. We start with $D_n^- = \tilde{d}_N e^{E_{0N}t} = d_N^- e^{E_{0N}t}$ and iterating this term in (3.7) we find a first family of solutions:

$$D_n^-(t) = e^{(E_{0N} - (n-N)\sigma_2)t} d_n^-, \quad \beta_{Nn} d_n^- = \sum_{q=N}^{n-2} \lambda_{qn} d_q^- \quad (\beta_{02} < 0 \text{ if } N = 0) \tag{B1a}$$

Second we use the integral equation (3.6) with integration d_n^- at $t = 0$. Of course now the integration is always possible. However two different cases occur depending on whether the integrand is an exponential of a constant, in the second case a term proportional to t appears. We find three different possibilities:

(i) If $d_0^- = d_1^- = \cdots d_{N-1}^- = 0$, $d_N^- \neq 0$, $d_{N+1}^- = 0$, $N = 0, 1, 2, \ldots$, $D_N^- = e^{E_{0N}t} d_N^-$, then for $n \geqslant N+2$ there exists a first family of solutions depending only on one parameter d_N^-:

$$D_n^-(t) = e^{(N_{0N} - (n-N)\sigma_2)t} d_n^-, \qquad \beta_{Nn} d_n^- = \sum_{q=N}^{n-2} \lambda_{qn} d_q^- \tag{B1b}$$

For $N = 0$, we further require that $\sigma(\theta)$ is such that $\beta_{0n} \neq 0$ and take advantage of the fact that the sequence β_{0n} is decreasing [Eqs. (3.3) and (3.5)], $\beta_{0,n} < 0$ for $n > n_0$ as soon as there exists n_0 with $\beta_{0,n_0} < 0$. We remark that (B1a, b) represent the same family of solutions but they are obtained with different assumptions on β_{0n} for $N = 0$.

There exists a family of solutions, called here (ii), which can equivalently be deduced from (3.6) or (3.7). It is obtained in (3.7) with $\tilde{d}_0 \neq 0$, $\tilde{d}_1 \neq 0$, $\beta_{01} = 0$ and it follows that $\beta_{02} < \theta$, but it can also be obtained from (3.6).

(ii) If $d_0^- \neq 0$, $d_1^- \neq 0$, $\beta_{01} = 0$, or $\sigma_2 = \tau_1 - \tau_3$, then there exists a second family of solutions depending on the two arbitrary parameters d_0^-, d_1^-:

$$D_n^-(t) = e^{(E_{00} - n\sigma_2)t} d_n^- \ \forall n, \qquad d_n^- \beta_{0,n} = \sum_{q=0}^{n-2} \lambda_{qn} d_q^-, \qquad n \geqslant 2 \tag{B2}$$

$\sigma_2 = \tau_1 - \tau_3$, $\beta_{0,n} = -(n-1)\tau_1 + n\tau_3 - \tau_{2n+1}$. The relation $d_1^- = -cd_0^-$ leads to the B.K.W. odd mode:

$$d_1^- = -Cd_0^- \to d_n^- = (-1)^n C^n d_0^- \text{ with the identity } \beta_{0n} C^n = \sum_{q=0}^{n-2} (-1)^q \lambda_{qn} C^q \tag{B2'}$$

(iii) If $d_0^- \neq 0$, $d_1^- = 0$, $d_N^- \neq 0$, $N \geqslant 2$, $\beta_{0N} = \tau_1 - \tau_{2N+1} - N\sigma_2 = 0$ then we find a two-parameter family of solutions

$$D_n^- = e^{(E_{00} - n\sigma_2)t}(d_n^- + t\tilde{\tilde{d}}_n); \quad d_0^- \text{ arbitrary}, \quad \tilde{\tilde{d}}_0 = 0; \quad 1 \leqslant n \leqslant N-1, \quad d_1^- = 0$$

$$\begin{gathered} d_n^- = \sum_{q=0}^{n-2} \frac{\lambda_{qn} d_q^-}{\beta_{0n}}, \quad \tilde{\tilde{d}}_n = 0; \qquad d_N^- \text{ atbitrary}, \quad \tilde{\tilde{d}}_n = \sum_{q=0}^{N-2} \lambda_{qn} d_q^-; \\ n = N+1, \quad d_{N+1}^- = \sum_0^{N-1} \frac{\lambda_{qN+1}}{\beta_{0,B+1}} d_q^- \end{gathered} \tag{B3}$$

$$\tilde{\tilde{d}}_{N+1} = 0; \qquad n \geqslant N+2, \quad d_n^- = \sum_0^{n-2} \frac{\lambda_{qn}}{\beta_{0,n}} d_q^- - \tilde{\tilde{d}}_n, \quad \tilde{\tilde{d}}_n = \sum_0^{n-2} \frac{\lambda_{qn}}{\beta_{0,n}} \tilde{\tilde{d}}_q$$

Here also due to $\beta_{0N} = 0$ we know that no other β_{0n} can vanish.

B3. Mixing Fundamental Solutions. The mixing of two fundamental solutions N_1 and N_2, $N_1 < N_2$, of class (i) with two arbitrary constant $d^-_{N_1}$, $d^-_{N_2}$ can be obtained as a particular case of the general formalism Eq. (3.9a, b):

$$D^-_n(t) = e^{E_{0n}t} a_{0n} + \sum_{N_1}^{n-2} a_{m+1,n} e^{[E_{0m} - (n-m)\sigma_2]t} \tag{B4a}$$

$$a_{m+1,n}\beta_{m,n} = \lambda_{m,n} a_{0m} + \sum_{q=m+2}^{n-2} \lambda_{qn} a_{m+1,q}, \qquad a_{0n} = d^-_n - \sum_{N_1}^{n-2} a_{m+1,n} \tag{B4b}$$

We find for the coefficients: $d^-_{N_1}$, $d^-_{N_2}$ arbitrary and

$$a_{0n} = 0 \text{ except } a_{0N_1} = d^-_{N_1}, \qquad a_{0N_2} = d^-_{N_2} - a_{N_1+1,N_2},$$
$$d^-_{N_1+1} = 0 \text{ if } N_2 > N_1 + 1 \tag{B5b}$$
$$a_{m,n} = 0 \text{ except } a_{N_i+1,n}\beta_{N_i,n} = \lambda_{N_i,n} a_{0N_i} + \sum_{q=N_i+2}^{n-2} \lambda_{qn} a_{N_i+1,q}, \qquad i = 1, 2$$

and for the moments

$$D^-_{N_1} = e^{E_{0N_1}t} d^-_{N_1}, \qquad D^-_n = e^{[E_{0N_1} - (n-N_1)\sigma_2]t} a_{N_1+1,n},$$
$$N_1 < n < N_2 + 1, \quad n \neq N_2 \tag{B5a}$$
$$D^-_{N_2} = e^{E_{0N_2}t} a_{0N_2} + e^{[E_{0N_1} - (N_2 - N_1)\sigma_2]t} a_{N_1+1,N_2}$$
$$D^-_n = e^{[E_{0N_2} - (n-N_2)\sigma_2]t} a_{N_2+1,n} + e^{[E_{0N_1} - (n-N_1)\sigma_2]t} a_{N_1+1,n}, \qquad n \geq N_2 + 2$$

For the proof we substitute the coefficients relations (B5b) into (B4b) and verify recursively that they hold.

B4. Examples of $\sigma(\theta)$ Models. We give simple examples of $\sigma(\theta)$ models corresponding to $\beta_{0n} \neq 0\ \forall n$ class (i), $\beta_{01} = 0$ class (ii), $\beta_{0N} = 0$ for some N integer > 1 class (iii).

Firstly we choose $\sigma(\theta)$ as a sum of δ distribution functions

$$\sigma(\theta) = \frac{1}{2} \sum_{i=1}^{M} \lambda_i [\delta(\theta - \theta_i) + \delta(\theta + \theta_i)], \qquad \sum_1^M \lambda_i = 1$$
$$\beta_{0,n} = \sum_{i=1}^{M} \lambda_i (1 - z_i^2)[z_i + z_i^3 + \cdots z_i^{2n-1} - n z_i^2],$$
$$|z_i| \neq 1, \neq 0, \cos\theta_i = z_i, \sigma_0 = 1 \tag{B6}$$

The simplest example is provided with $M = 1$ and only one $\theta_i = \theta_1$ value for which $\lambda_1 = 1$ and $\beta_{0,n} = (1 - z_1^2)\ [z_1 + z_1^3 + \cdots z_1^{2n-1} - n z_1^2]$. If we choose $z_1 < 0$ then $\beta_{0,n} < 0\ \forall n$ and $\sigma(\theta)$ is of class (i). We notice that $\beta_{0,1} =$

$(1-z_1^2)\, z_1(1-z_1) \neq 0$ and in this family of $\sigma(\theta)$ cross sections we have no example of class (ii). Similarly $\beta_{02} = z_1(1-z_1^2)\,(1-z_1)^2 \neq 0$ and in class (iii) we cannot obtain here $\sigma(\theta)$ corresponding to $\beta_{02}=0$. For $n \geqslant 3$, if we except the trivial $z_1=0$ zero, β_{0n} has always only one zero $z_{1,n}$: $z_{1,3} \simeq 0.3926$, $z_{1,4} \simeq 0.2696$, $z_{1,5} \simeq 0.2091$ which tends to 0^+ when $n \to \infty$. With $M=2$ in Eq. (B.6) and two θ_i values, θ_1 and θ_2, we have more freedom in order to construct σ models belonging to classes (i) and (ii). However the positivity of σ requires $\lambda_1 > 0$ and $\lambda_2 > 0$. From the conditions $\beta_{0n}=0$, $\sigma_0 = 1$:

$$\lambda_1 = -\frac{N_2}{N_1 - N_2}, \qquad \lambda_2 = \frac{N_1}{N_1 - N_2},$$
$$N_i = (1-z_i^2)(z_i + z_i^3 + \cdots + z_i^{2n-1} - nz_i^2), \qquad i = 1, 2 \tag{B7a}$$

$\sigma(\theta) > 0$ is satisfied if z_1 and z_2 are such that $N_1 > 0$ and $N_2 < 0$.

For simplicity we restrict the discussion to $z_2 = -z_1$ and obtain

$$\lambda_1 = \frac{z_1 + z_1^3 + \cdots z_1^{2n-1} + nz_1^2}{2(z_1 + z_1^3 + \cdots z_1^{2n-1})}, \qquad \lambda_2 = \frac{z_1 + z_1^3 + \cdots z_1^{2n-1} - nz_1^2}{2(z_1 + z_1^3 + \cdots z_1^{2n-1})},$$
$$\beta_{0,n} = 0 \tag{B7b}$$

If $\beta_{0,1}=0$, we find $\lambda_1 = (1+z_1)/2 > 0$, $\lambda_2 = (1-z_1)/2 > 0$, we have a model for class (ii), for instance[9] for the B.K.W. odd mode.

If $\beta_{0,2}=0$, we find $\lambda_1 = (1+z_1)^2/2(1+z_1^2) > 0$, $\lambda_2 = (1-z_1)^2/2(1+z_1^2) > 0$ and we find a model of class (iii).

Secondly we consider smooth $\sigma(\theta)$ models: $\sigma = (1/4)\cos(\theta/2)\, \sum_1^M \bar{\lambda}_m(\cos\theta)^{m-1}$. For $M=1$, $\bar{\lambda}_1 = 1$ we find a model of class (i):

$$\sigma(\theta) = \frac{1}{4}\cos\frac{\theta}{2}, \qquad \sigma_0 = 1, \qquad \beta_{0,1} = -\frac{16}{315}, \qquad \beta_{0,n} < 0, \qquad n \geqslant 1 \tag{B8}$$

For $M=2$ we give a model[7] of class (ii):

$$\sigma = \frac{57}{4(68)}\cos\frac{\theta}{2}\left(1 + \frac{11}{19}\cos\theta\right), \qquad \sigma_0 = 1, \qquad \beta_{0,1} = 0, \qquad \beta_{0,n} < 0, \qquad n > 1 \tag{B9}$$

For $M=3$ we find models of class (iii) with $\beta_{0,2}=0$, $\beta_{0,n}<0$ for $n>2$:

$$\sigma = \frac{1}{4}\cos\frac{\theta}{2}\,(\bar{\lambda}_1 + \bar{\lambda}_2\cos\theta + \bar{\lambda}_2\cos^2\theta), \qquad \sigma_0 = 1, \qquad \beta = 0,$$
$$\bar{\lambda}_1 \in [0.14, 0.56] \to \sigma(\theta) > 0 \tag{B10}$$

$$\frac{\bar{\lambda}_2}{3} + \frac{7}{15}\bar{\lambda}_3 = 1 - \bar{\lambda}_1, \qquad (31)(32)\frac{\bar{\lambda}_2}{3} - 177\bar{\lambda}_3 = (13()32)\bar{\lambda}_1$$

APPENDIX C

We want to obtain upper bounds on

$$N_2(t)=\sum_0^\infty |D_n^-(t)|^2\,\lambda_n, \qquad \lambda_n=\frac{\Gamma(n+3/2)}{\Gamma(n+1)} \tag{C1}$$

C1: Bound for λ_n: We obtain the following bound

$$\lambda_n<\lambda_m\lambda_{n-m-2}\Lambda, \qquad \Lambda=15[\Gamma(\tfrac{1}{2})]^{-1}, \qquad m\in[0,n-2], \qquad n\geqslant 2 \tag{C2}$$

We can write

$$\lambda_n(\lambda_m\lambda_{n-m-2})^{-1}=(2n+1)!!\,[n!\,\Gamma(\tfrac{3}{2})\beta_n^m]$$

$$\beta_n^m=\frac{(2m+1)!!\,[2(n-m)-3]!!}{m!(n-m-2)!}=\beta_n^{n-m-2}$$

We notice that for n fixed and $p<(n-1)/2$, $p<(n-1)/2$, β_n^p is increasing:

$$\beta_n^p-\beta_n^{p+1}=\left\{\frac{(2p+1)!!\,[2(n-p)-5]!!}{(p+1)!\,(n-p-2)!}\right\}(2p+1-n)$$

It follows that $\beta_n^m>\beta_n^0$ and finally for $n\geqslant 2$

$$\frac{\lambda_n}{\lambda_m\lambda_{n-m-2}}\leqslant\frac{\lambda_n}{\lambda_0\lambda_{n-2}}=\frac{(4n^2-1)(n-2)!}{(n^2-n)\,\Gamma(\tfrac{3}{2})(2n-3)!!}\leqslant\Lambda$$

C2: Existence Proof for the Fundamental Solutions of Class (i): From the definition we have $\beta_{Nn}\neq 0$ for $N\neq 0$ and $|D_n^-(t)|<|d_n^-|$ and consequently $N_2(t)<N_2(0)$. We seek the conditions on $\sigma(\theta)$ and c (constant entering into the definition of the B.K.W. even mode) in order that $N_2(0)<\infty$. We recall

$$d_N^-\neq 0, \qquad d_{N+1}^-=0, \qquad d_n^-=\sum_{q=N}^{n-2}\frac{\lambda_{qn}}{\beta_{Nn}}\,d_q^-, \qquad n\geqslant N+2,$$

$$D_n^-(t)=d_n^-\,e^{[E_{0N}-(n-N)\sigma_2]t}\,d_n^- \tag{C3}$$

where N is 0 or and integer. We define $\beta=\inf_n|\beta_{Nn}|$, $\beta=|\beta_{NN+2}|\neq 0$ for $N\neq 0$,

$$A_0(\beta,\sigma,c)=\beta^2-\frac{15c^4}{(1-c^2)^3}(1+4c^2+c^4)\left[\int_{-\pi}^{+\pi}\sigma(\theta)\,|\cos\theta|\,d\theta\right]^2,$$

$$A_0>0 \text{ if } \beta \text{ fixed and } |c| \text{ small} \tag{C4}$$

We will show the following result:

$$N_2(0) < \beta^2\lambda_N(d_N^-)^2/A_0 \text{ if} \int \sigma|\cos\theta|\,d\theta < \infty, \qquad A_0 > 0, \beta \neq 0 \tag{C5}$$

Using the definitions of β, λ_{qn} we have

$$\beta|d_n^-| < \sum_{q=N}^{n-2} (n-q-1)|C|^{n-q} C_n^q |E_{n-q,n}|\,|d_q^-|$$

We multiply both sides by $(\lambda_n)^{1/2}$, introduce at the right-hand side the inequality (C2) for λ_n, then take the square of both sides and apply the Schwarz inequality:

$$\lambda_n\beta^2 d_n^{-2} < A \sum_{q=N} \lambda_q\lambda_{n-q-2}(n-q-1)^2 c^{2(n-q)}(d_q^-)^2 B_0 \tag{C6}$$

$$B_0 = \sum_{q=N}^{n-2} (C_n^q|E_{n-qn}|)^2 < \left(\sum_{N}^{n-2} C_n^q|E_{n-qn}|\right)^2$$

$$B_0 < \left[\int \sigma|\cos\theta| \sum \cos\theta^{2q}\sin\theta^{2(n-q)}C_n^q\,d\theta\right]^2 < \left[\int \sigma|\cos\theta|\,d\theta\right]^2$$

For all $n \geqslant N+2$ we sum (C6) adding at the end $\beta^2\lambda_n(d_N^-)^2$ in both sides and find

$$N_2(t=0) < \beta^2\lambda_N(d_N^-)^2 + A\left[\int \sigma|\cos\theta|\,d\theta\right]^2 B_1$$

$$B_1 = \sum_{n=2}^{\infty} \sum_{p+q=n-1} \lambda_q(d_q^-)^2(\lambda_{p-1}p^2c^{2(p+1)}) = N_2(0)\sum_{p=1}^{\infty} \lambda_{p-1}p^2c^{2(p+1)} \tag{C7}$$

From the definition (C1) of λ_p we find $\lambda_{p-1} < p\Gamma(\frac{1}{2})$ and

$$\sum_{1}^{\infty} p^3c^{2(p+1)} = \frac{c^4(1+4c^2+c^4)}{(1-c^2)^3} \qquad \text{if} \quad |c|<1$$

Subtracting the term proportional to B_1 in both sides of (C7) we obtain $N_2A_0 < \beta^2\lambda_N(d_N^-)^2$ with A_0 defined in (C4) and finally the result (C5) if $A_0 > 0$.

C3: Existence Proof for the General Odd B.K.W. Solutions. We start with the differential system (24c) that we multiply by $\lambda_n D_n^-(t)$ and integrate from $t=0$ to t

$$\lambda_n(D_n^-(t))^2 = e^{2E_{0n}t}\lambda_n(d_n^-)^2 + 2\int_0^t e^{2E_{0n}(t-t')}\lambda_n D_n^-(t')\,A_1(t')\,dt'$$

$$A_1(t) = \sum_{q=0}^{n-2} (n-q-1)[\omega(t)]^{n-q} D_q^-(t) C_n^q E_{n-qn}, \tag{C8}$$

$$\omega = ce^{-\sigma_2 t}, \qquad -E_{0n} = 1 - \tau_{2n+1} > 0$$

and we shall obtain two main results

$$N_2(t) < \infty \quad \text{if} \quad N_2(0) < \infty,\ |c| < 1 \text{ and } \int \sigma(\theta)\,|\cos\theta|\,d\theta < \infty \tag{C9}$$

$$\underset{t\to\infty}{N_2(t) \to 0} \quad \text{if} \quad \text{further } \gamma = \inf_{\forall n}(-E_{0n}) > 0 \tag{C10}$$

For these results we shall obtain, at an intermediate stage, an integral inequality

$$\left.\begin{aligned} &N_2(t)e^{2\gamma t} < M(t) \\ &\quad M(t) = N_2(0) + 2(15)^{1/2}\int \sigma\,|\cos\theta|\,d\theta \int_0^t e^{2\gamma t'} N_2(t')\,H(t')\,dt' \\ &\quad H(t) = \omega^2(t)[1+\omega(t)]/[1-\omega(t)]^3 \end{aligned}\right\} \tag{C11}$$

First we notice that $|A_1| < \sum (n-q-1)\,|\omega|^{n-q}\,|D_q^-|\,\sum C_n^q |E_{n-qn}|$ where as in Section C2 the last sum is bounded by $\int \sigma\,|\cos\theta|\,d\theta$. We start from (C8), take the modulus of both sides, use the bound (C2) for λ_n, and sum over n. We find

$$\begin{aligned} &N_2(t) < e^{-2\gamma t}N_2(0) + 2\Lambda^{1/2}\int \sigma\,|\cos\theta|\,d\theta \int_0^t e^{-2\gamma(t-t')}\,A_2(t')\,dt' \\ &A_2(t) = \sum_{n=2}^{\infty} |D_n^-|\,\lambda_n^{1/2} \sum_{p+q=n-1} \lambda_q^{1/2}|D_q^-|\,\lambda_{p-1}^{1/2}\,p\omega^{p+1} \\ &\quad A_2 = \sum_1^{\infty} \lambda_{p-1}^{1/2}\,p\omega^{p+1} \sum_{q=0}^{\infty} |D_q^-\lambda_q^{1/2}|\,|D_{p+q+1}^-\lambda_{p+q}^{1/2}| < \sum \lambda_{p-1}^{1/2}\,p\omega^{p+1}N_2 \end{aligned} \tag{C12}$$

using the Schwarz inequality. We bound λ_{p-1} by $p^2\Gamma(\frac{1}{2})$ and (C11–C12) become identical if $|\omega| < 1$ and $H = \sum p^2\omega^{p+1}$. Second we notice that (C11) is equivalent to the differential inequality $dM/dt < M[H(t)\,2(15)^{1/2}\int \sigma\,|\cos\theta|\,d\theta]$ that we integrate and finally find for $N_2(t)$:

$$N_2(t) < e^{-2\gamma t}\left(N_2(0)\exp\left\{2(15)^{1/2}\int_0^t \frac{\omega^2(t')[1+\omega(t')]\,dt'}{[1-\omega(t')]^3}\right\}\right) \tag{C13}$$

Because the integral exists when $t \to \infty$, we can put infinity as the upper limit of integration and obtain (C9), (C10).

APPENDIX D

$-(B_{0n}+B_{nn})=\int_{-\pi}^{+\pi}\sigma(\theta)\,b_n(z)\,d\theta$, $z=\cos\theta$, $b_n=1-z^{2n}-(1-z^2)^n$, $n=1$, is a positive increasing sequence. It is sufficient to notice that: $b_{n+1}-b_n=z^{2n}(1-z^2)+z^2(1-z^2)^n\geqslant 0$, for $|z|\leqslant 1$.

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